

SECOND-ORDER NECESSARY CONDITIONS IN PONTRYAGIN FORM FOR OPTIMAL CONTROL PROBLEMS*

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Abstract. In this article, we state and prove first- and second-order necessary conditions in Pontryagin form for optimal control problems with pure state and mixed control-state constraints. We say that a Lagrange multiplier of an optimal control problem is a Pontryagin multiplier if it is such that Pontryagin's minimum principle holds, and we call optimality conditions in Pontryagin form those which only involve Pontryagin multipliers. Our conditions rely on a technique of partial relaxation, and apply to Pontryagin local minima.

Key words. optimal control, pure state and mixed control-state constraints, Pontryagin's principle, Pontryagin multipliers, second-order necessary conditions, partial relaxation

AMS subject classifications. 49K15, 34K35, 90C48

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1. Introduction. The optimization theory in Banach spaces, in particular, optimality conditions of order one [29, 32] and two [12, 23, 24], applies to optimal control problems. With this approach, constraints of various kinds can be considered, and optimality conditions are derived for weak local minima of optimal control problems. Second-order necessary and sufficient conditions are thereby obtained by Stefani and Zezza [31] in the case of mixed control-state equality constraints, or by Bonnans and Hermant [6] in the case of pure state and mixed control-state constraints. These optimality conditions always involve Lagrange multipliers.

Another class of optimality conditions, necessary and of order one, for optimal control problems comes from Pontryagin's minimum principle. Formulated in the historical book [27] for basic problems, including first-order pure state constraints, this principle has then been extended by many authors. Mixed control-state constraints enter, for example, the framework developed by Hestenes [21], whereas pure state, and later pure state and mixed control-state, constraints are treated in early Russian references such as the works of Dubovitskii and Milyutin [16, 17], as highlighted by Dmitruk [13]. Let us mention the survey by Hartl, Sethi, and Vickson [19] and its bibliography for more references on Pontryagin's principles.

Second-order optimality conditions are said in this article to be in Pontryagin form if they only involve Lagrange multipliers for which Pontryagin's minimum principle holds. This restriction to a subset of multipliers is a challenge for necessary conditions, and enables sufficient conditions to give strong local minima. To our knowledge, such conditions have been stated for the first time, under the name of quadratic conditions, for problems with mixed control-state equality constraints by Milyutin

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and Osmolovskii [25]. Proofs are given by Osmolovskii and Maurer [26], under a restrictive full-rank condition for the mixed equality constraints, that could not, for instance, be satisfied by pure state constraints.

The main novelty of this paper is to provide second-order necessary conditions in Pontryagin form for optimal control problems with pure state and mixed control-state constraints. We use the same technique as Dmitruk in his derivation of Pontryagin's principle for a general optimal control problem [13]: also, a partial relaxation of the problem, based on the sliding modes introduced by Gamkrelidze [18]. These convexifications of the set of admissible velocities furnish a sequence of auxiliary optimal control problems, and at the limit, necessary conditions appear to be in Pontryagin form. We thereby get our own version of Pontryagin's minimum principle, as first-order necessary conditions. Then, combining the partial relaxation with a reduction approach [5, 22] and a density argument [2], we obtain second-order necessary conditions in Pontryagin form for a Pontryagin local minimum of our problem. This technique requires us to consider a variant of the previous auxiliary problems, but not to compute any envelope-like effect of Kawasaki [23]. Another result that is worth mentioning is the second-order necessary conditions for a local solution of an abstract optimization problem, that we apply to the partially relaxed problems. We derive them directly on a large set of directions in L^2 , which then simplifies the density argument, compared with [2], and avoid a flaw that was mentioned in the proof of the density result in [6].

Second-order sufficient conditions for strong local minima of similar optimal control problems constitute another work by the authors of this paper [4]. They rely on an extension of the decomposition principle of Bonnans and Osmolovskii [7], and on the reduction approach; quadratic growth for a strong local minimum is characterized.

The paper is organized as follows. In section 2, we set our optimal control problem and define various notions of multipliers and of minima. Section 3 is devoted to the first-order necessary conditions: they are stated, under the form of Pontryagin's minimum principle, in section 3.1; our partial relaxation approach is detailed in section 3.2, and then used to prove the first-order conditions in section 3.3. Section 4 is devoted to the second-order necessary conditions: they are stated in section 4.1, and proved in section 4.2 by partial relaxation combined with reduction and density. We have postponed our abstract optimization results to Appendix A, a qualification condition to Appendix B, and an example about Pontryagin's principle to Appendix C.

Notation. For a function h that depends only on time t , we denote by h_t its value at time t , by $h_{i,t}$ the value of its i th component if h is vector valued, and by \dot{h} its derivative. For a function h that depends on (t, x) , we denote by $D_t h$ and $D_x h$ its partial derivatives. We use the symbol D without any subscript for the differentiation w.r.t. all variables except t , e.g., $Dh = D_{(u,y)}h$ for a function h that depends on (t, u, y) . We use the same convention for higher-order derivatives.

We identify the dual space of \mathbb{R}^n with the space \mathbb{R}^{n*} of n -dimensional horizontal vectors. Generally, we denote by X^* the dual space of a topological vector space X . Given a convex subset K of X and a point x of K , we denote by $T_K(x)$ and $N_K(x)$ the tangent and normal cone to K at x , respectively; see [9, section 2.2.4] for their definition.

We denote by $|\cdot|$ both the Euclidean norm on finite-dimensional vector spaces and the cardinal of finite sets, and by $\|\cdot\|_s$ and $\|\cdot\|_{q,s}$ the standard norms on the Lebesgue spaces L^s and the Sobolev spaces $W^{q,s}$, respectively.

We denote by $BV([0, T])$ the space of functions of bounded variation on the closed interval $[0, T]$. Any $h \in BV([0, T])$ has a derivative dh which is a finite Radon measure

on $[0, T]$ and h_0 (resp., h_T) is defined by $h_0 := h_{0+} - dh(0)$ (resp., $h_T := h_{T-} + dh(T)$). Thus $BV([0, T])$ is endowed with the following norm: $\|h\|_{BV} := \|dh\|_{\mathcal{M}} + |h_T|$. See [1, section 3.2] for a rigorous presentation of BV .

All vector-valued inequalities have to be understood coordinatewise.

2. Setting.

2.1. The optimal control problem. Consider the *state equation*

$$(2.1) \quad \dot{y}_t = f(t, u_t, y_t) \quad \text{for a.a. } t \in (0, T).$$

Here, u is a *control* which belongs to \mathcal{U} , y is a *state* which belongs to \mathcal{Y} , where

$$(2.2) \quad \mathcal{U} := L^\infty(0, T; \mathbb{R}^m), \quad \mathcal{Y} := W^{1,\infty}(0, T; \mathbb{R}^n),$$

and $f: [0, T] \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the *dynamics*. Consider constraints of various types on the system: the *mixed control-state constraints*, or mixed constraints

$$(2.3) \quad c(t, u_t, y_t) \leq 0 \quad \text{for a.a. } t \in (0, T),$$

the *pure state constraints*, or state constraints

$$(2.4) \quad g(t, y_t) \leq 0 \quad \text{for a.a. } t \in (0, T),$$

and the *initial-final state constraints*

$$(2.5) \quad \begin{cases} \Phi^E(y_0, y_T) = 0, \\ \Phi^I(y_0, y_T) \leq 0. \end{cases}$$

Here $c: [0, T] \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^{n_c}$, $g: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n_g}$, $\Phi^E: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n_{\Phi^E}}$, $\Phi^I: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n_{\Phi^I}}$. Consider finally the *cost function* $\phi: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$. The *optimal control problem* is then

$$(P) \quad \min_{(u,y) \in \mathcal{U} \times \mathcal{Y}} \phi(y_0, y_T) \quad \text{subject to} \quad (2.1)\text{--}(2.5).$$

2.2. Definitions and assumptions. Similarly to [31, Definition 2.1], we introduce the following Carathéodory-type regularity notion.

DEFINITION 2.1. A function $\varphi: [0, T] \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^s$ is uniformly quasi- C^k iff

1. for a.a. t , $(u, y) \mapsto \varphi(t, u, y)$ is of class C^k , and the modulus of continuity of $(u, y) \mapsto D^k \varphi(t, u, y)$ on any compact of $\mathbb{R}^m \times \mathbb{R}^n$ is uniform w.r.t. t ;
2. for all $(u, y) \in \mathbb{R}^m \times \mathbb{R}^n$, $t \mapsto \varphi(t, u, y)$ is measurable and, for $j = 0$ to k , $t \mapsto D^j \varphi(t, u, y)$ is essentially bounded.

Remark 2.2. If φ is uniformly quasi- C^k , then $D^j \varphi$ for $j = 0, \dots, k$ are essentially bounded on any compact, and $(u, y) \mapsto D^j \varphi(t, u, y)$ for $j = 0, \dots, k - 1$ are locally Lipschitz, uniformly w.r.t. t . In particular, if f is uniformly quasi- C^1 , then by the Cauchy–Lipschitz theorem, for any $(u, y^0) \in \mathcal{U} \times \mathbb{R}^n$, there exists at most one $y \in \mathcal{Y}$ such that (2.1) holds and $y_0 = y^0$; we denote it by $y[u, y^0]$ when it exists.

The minimal regularity assumption through all the paper is the following.

Assumption 1. The mappings f , c , and g are uniformly quasi- C^1 , g is continuous, Φ^E , Φ^I , and ϕ are C^1 .

We call a *trajectory* any pair $(u, y) \in \mathcal{U} \times \mathcal{Y}$ such that (2.1) holds. We say that a trajectory is *feasible* for problem (P) if it satisfies constraints (2.3)–(2.5), and denote

by $F(P)$ the set of feasible trajectories. We define the *Hamiltonian* and the *augmented Hamiltonian*, respectively, by

$$(2.6) \quad H[p](t, u, y) := pf(t, u, y), \quad H^a[p, \nu](t, u, y) := pf(t, u, y) + \nu c(t, u, y)$$

for $(p, \nu, t, u, y) \in \mathbb{R}^{n^*} \times \mathbb{R}^{n_{c^*}} \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^n$, and the *endpoints Lagrangian* by

$$(2.7) \quad \Phi[\beta, \Psi](y_0, y_T) := \beta\phi(y_0, y_T) + \Psi\Phi(y_0, y_T)$$

for $(\beta, \Psi, y_0, y_T) \in \mathbb{R} \times \mathbb{R}^{n_{\Phi^*}} \times \mathbb{R}^n \times \mathbb{R}^n$, where $n_{\Phi} = n_{\Phi^E} + n_{\Phi^I}$ and $\Phi = \begin{pmatrix} \Phi^E \\ \Phi^I \end{pmatrix}$.

We denote by $c(\cdot, u, y) : t \mapsto c(t, u_t, y_t)$, $g(\cdot, y) : t \mapsto g(t, y_t)$ when $(u, y) \in \mathcal{U} \times \mathcal{Y}$,

$$(2.8) \quad K_c := L^\infty(0, T; \mathbb{R}^{n_c}), \quad K_g := C([0, T]; \mathbb{R}^{n_g}), \quad K_{\Phi} := \{0\}_{\mathbb{R}^{n_{\Phi^E}}} \times \mathbb{R}^{n_{\Phi^I}},$$

so that the constraints (2.3)–(2.5) can be rewritten as

$$(2.9) \quad c(\cdot, u, y) \in K_c, \quad g(\cdot, y) \in K_g, \quad \Phi(y_0, y_T) \in K_{\Phi}.$$

Recall that the dual space of $C([0, T]; \mathbb{R}^{n_g})$ is the space $\mathcal{M}([0, T]; \mathbb{R}^{n_{g^*}})$ of finite vector-valued Radon measures. We denote by $\mathcal{M}([0, T]; \mathbb{R}^{n_{g^*}})_+$ the cone of positive measures in this dual space. Let

$$(2.10) \quad E := \mathbb{R} \times \mathbb{R}^{n_{\Phi^*}} \times L^\infty(0, T; \mathbb{R}^{n_{c^*}}) \times \mathcal{M}([0, T]; \mathbb{R}^{n_{g^*}})$$

and let $\|\cdot\|_E$ be defined, for any $\lambda = (\beta, \Psi, \nu, \mu) \in E$, by

$$(2.11) \quad \|\lambda\|_E := |\beta| + |\Psi| + \|\nu\|_1 + \|\mu\|_{\mathcal{M}}.$$

Let $(\bar{u}, \bar{y}) \in F(P)$. By a slight abuse of notation we denote by N_{K_c} the set of elements in the normal cone to K_c at $c(\cdot, \bar{u}, \bar{y})$ that belong to $L^\infty(0, T; \mathbb{R}^{n_{c^*}})$ (instead of the larger set $L^\infty(0, T; \mathbb{R}^{n_{c^*}})^*$), i.e.,

$$(2.12) \quad N_{K_c}(c(\cdot, \bar{u}, \bar{y})) := \{ \nu \in L^\infty(0, T; \mathbb{R}_+^{n_{c^*}}) : \nu_t c(t, \bar{u}_t, \bar{y}_t) = 0 \text{ for a.a. } t \}.$$

Let N_{K_g} be the normal cone to K_g at $g(\cdot, \bar{y})$, i.e.,

$$(2.13) \quad N_{K_g}(g(\cdot, \bar{y})) := \left\{ \mu \in \mathcal{M}([0, T]; \mathbb{R}^{n_{g^*}})_+ : \int_{[0, T]} (d\mu_t g(t, \bar{y}_t)) = 0 \right\}.$$

Let $N_{K_{\Phi}}$ be the normal cone to K_{Φ} at $\Phi(\bar{y}_0, \bar{y}_T)$, i.e.,

$$(2.14) \quad N_{K_{\Phi}}(\Phi(\bar{y}_0, \bar{y}_T)) := \left\{ \Psi \in \mathbb{R}^{n_{\Phi^*}} : \begin{array}{l} \Psi_i \geq 0 \\ \Psi_i \Phi_i(\bar{y}_0, \bar{y}_T) = 0 \end{array} \text{ for } n_{\Phi^E} < i \leq n_{\Phi} \right\}.$$

Finally, let

$$(2.15) \quad N(\bar{u}, \bar{y}) := \mathbb{R}_+ \times N_{K_{\Phi}}(\Phi(\bar{y}_0, \bar{y}_T)) \times N_{K_c}(c(\cdot, \bar{u}, \bar{y})) \times N_{K_g}(g(\cdot, \bar{y})) \subset E.$$

We denote

$$(2.16) \quad \mathcal{P} := BV([0, T]; \mathbb{R}^{n^*}).$$

Given $(\bar{u}, \bar{y}) \in F(P)$ and $\lambda = (\beta, \Psi, \nu, \mu) \in E$, we consider the *costate equation* in \mathcal{P}

$$(2.17) \quad \begin{cases} -dp_t = D_y H^a[p_t, \nu_t](t, \bar{u}_t, \bar{y}_t)dt + d\mu_t Dg(t, \bar{y}_t), \\ p_T = D_{y_T} \Phi[\beta, \Psi](\bar{y}_0, \bar{y}_T). \end{cases}$$

We have the following well-known result.

LEMMA 2.3. *Let $(\bar{u}, \bar{y}) \in F(P)$. For any $\lambda \in E$, there exists a unique solution of the costate equation (2.17), that we denote by p^λ . The mapping*

$$(2.18) \quad \lambda \in E \mapsto p^\lambda \in \mathcal{P}$$

is linear continuous.

DEFINITION 2.4. *Let $(\bar{u}, \bar{y}) \in F(P)$ and $\lambda = (\beta, \Psi, \nu, \mu) \in E$. We say that the solution of the costate equation (2.17) $p^\lambda \in \mathcal{P}$ is an associated costate iff*

$$(2.19) \quad -p_0^\lambda = D_{y_0} \Phi[\beta, \Psi](\bar{y}_0, \bar{y}_T).$$

Let $N_\pi(\bar{u}, \bar{y})$ be the set of nonzero $\lambda \in N(\bar{u}, \bar{y})$ having an associated costate.

Let $(\bar{u}, \bar{y}) \in F(P)$. We define the set-valued mapping $U: [0, T] \rightrightarrows \mathbb{R}^m$ by

$$(2.20) \quad U(t) := \text{cl} \{u \in \mathbb{R}^m : c(t, u, \bar{y}_t) < 0\} \quad \text{for a.a. } t,$$

where cl denotes the closure in \mathbb{R}^m .

DEFINITION 2.5. *Let $(\bar{u}, \bar{y}) \in F(P)$. We say that the inward condition for the mixed constraints holds iff there exist $\gamma > 0$ and $\bar{v} \in \mathcal{U}$ such that*

$$(2.21) \quad c(t, \bar{u}_t, \bar{y}_t) + D_u c(t, \bar{u}_t, \bar{y}_t) \bar{v}_t \leq -\gamma, \quad \text{for a.a. } t.$$

Remark 2.6. If the inward condition holds, then there exists $\delta > 0$ such that, for a.a. t ,

$$(2.22) \quad B_\delta(\bar{u}_t) \cap U(t) = B_\delta(\bar{u}_t) \cap \{u \in \mathbb{R}^m : c(t, u, \bar{y}_t) \leq 0\},$$

where $B_\delta(\bar{u}_t)$ is the open ball in \mathbb{R}^m of center \bar{u}_t and radius δ . In particular, $\bar{u}_t \in U(t)$ for a.a. t .

In what follows, we will always make the following assumption.

Assumption 2. The inward condition for the mixed constraints holds.

We define now the notions of multipliers that we will consider. Recall that $N_\pi(\bar{u}, \bar{y})$ has been introduced in Definition 2.4.

DEFINITION 2.7. *Let $(\bar{u}, \bar{y}) \in F(P)$.*

(i) *We say that $\lambda \in N_\pi(\bar{u}, \bar{y})$ is a generalized Lagrange multiplier iff*

$$(2.23) \quad D_u H^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t) = 0 \quad \text{for a.a. } t.$$

We denote by $\Lambda_L(\bar{u}, \bar{y})$ the set of generalized Lagrange multipliers.

(ii) *We say that $\lambda \in \Lambda_L(\bar{u}, \bar{y})$ is a generalized Pontryagin multiplier iff*

$$(2.24) \quad H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) \leq H[p_t^\lambda](t, u, \bar{y}_t) \quad \text{for all } u \in U(t) \quad \text{for a.a. } t.$$

We denote by $\Lambda_P(\bar{u}, \bar{y})$ the set of generalized Pontryagin multipliers.

(iii) *We say that $\lambda \in \Lambda_P(\bar{u}, \bar{y})$ is a degenerate Pontryagin equality multiplier iff $\lambda = (\beta, \Psi, \nu, \mu)$ with $\Psi = (\Psi^E, \Psi^I)$ is such that $(\beta, \Psi^I, \nu, \mu) = 0$ and if equality holds in (2.24). We denote by $\Lambda_P^D(\bar{u}, \bar{y})$ the set of such multipliers.*

Remark 2.8.

1. The sets $\Lambda_L(\bar{u}, \bar{y})$, $\Lambda_P(\bar{u}, \bar{y})$, and $\Lambda_P^D(\bar{u}, \bar{y})$ are cones of nonzero elements, possibly empty, and $\Lambda_P^D(\bar{u}, \bar{y})$ is symmetric.
2. Assumption 2 will be needed to get that the component ν of a multiplier, associated with the mixed constraints, belongs to $L^\infty(0, T; \mathbb{R}^{n_c^*})$ and not only to $L^\infty(0, T; \mathbb{R}^{n_c})^*$. See [7, Theorem 3.1] and Theorem A.7 in Appendix A.
3. Let $\lambda \in \Lambda_P(\bar{u}, \bar{y})$. If Assumption 2 holds, then by Remark 2.6, \bar{u}_t is a local solution of the finite-dimensional optimization problem

$$(2.25) \quad \min_{u \in \mathbb{R}^m} H[p_t^\lambda](t, u, \bar{y}_t) \quad \text{subject to} \quad c(t, u, \bar{y}_t) \leq 0,$$

and ν_t is an associated Lagrange multiplier, for a.a. t .

4. See Appendix C for an example where there exists a multiplier such that (2.24) holds for all $u \in U(t)$, but not for all $u \in \{u \in \mathbb{R}^m : c(t, u, \bar{y}_t) \leq 0\}$.

We finish this section with various notions of minima, following [25].

DEFINITION 2.9. *We say that $(\bar{u}, \bar{y}) \in F(P)$ is a global minimum iff*

$$(2.26) \quad \phi(\bar{y}_0, \bar{y}_T) \leq \phi(y_0, y_T) \quad \text{for all } (u, y) \in F(P),$$

a Pontryagin minimum iff for any $R > \|\bar{u}\|_\infty$, there exists $\varepsilon > 0$ such that

$$(2.27) \quad \begin{aligned} \phi(\bar{y}_0, \bar{y}_T) \leq \phi(y_0, y_T) \quad \text{for all } (u, y) \in F(P) \text{ such that} \\ \|u - \bar{u}\|_1 + \|y - \bar{y}\|_\infty \leq \varepsilon \text{ and } \|u\|_\infty \leq R, \end{aligned}$$

a weak minimum iff there exists $\varepsilon > 0$ such that

$$(2.28) \quad \begin{aligned} \phi(\bar{y}_0, \bar{y}_T) \leq \phi(y_0, y_T) \quad \text{for all } (u, y) \in F(P) \text{ such that} \\ \|u - \bar{u}\|_\infty + \|y - \bar{y}\|_\infty \leq \varepsilon. \end{aligned}$$

Remark 2.10. Obviously, (2.26) \Rightarrow (2.27) \Rightarrow (2.28). Conversely, if (\bar{u}, \bar{y}) is a weak minimum for problem (P) , then it is a Pontryagin minimum for the problem obtained by adding the control constraint $|u_t - \bar{u}_t| \leq \varepsilon$, and a global minimum for the problem obtained by adding the same control constraint and the state constraint $|y_t - \bar{y}_t| \leq \varepsilon$.

3. First-order conditions in Pontryagin form.

3.1. Pontryagin’s minimum principle. First-order necessary conditions in Pontryagin form consist in proving the existence of Pontryagin multipliers. See Definitions 2.7 and 2.9 for the notions of multipliers and of minima. Our version of the well-known *Pontryagin’s principle* follows, and is proved in section 3.3. See [13] for a variant with the same approach, and [19] for a survey of this principle.

THEOREM 3.1. *Let (\bar{u}, \bar{y}) be a Pontryagin minimum for problem (P) and let Assumptions 1–2 hold. Then the set of generalized Pontryagin multipliers $\Lambda_P(\bar{u}, \bar{y})$ is nonempty.*

By Remark 2.10, we get the following.

COROLLARY 3.2. *Let (\bar{u}, \bar{y}) be a weak minimum for problem (P) and let Assumptions 1–2 hold. Then there exist $\varepsilon > 0$ and $\lambda \in \Lambda_L(\bar{u}, \bar{y})$ such that*

$$(3.1) \quad \left\{ \begin{array}{l} \text{for a.a. } t, \text{ for all } u \in U(t) \text{ such that } |u - \bar{u}_t| \leq \varepsilon, \\ H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) \leq H[p_t^\lambda](t, u, \bar{y}_t). \end{array} \right.$$

Proof. The extra control constraint $|u - \bar{u}_t| \leq \varepsilon$ for a.a. t is never active, therefore the set of Lagrange multipliers is unchanged. The set of Pontryagin multipliers is the set of Lagrange multipliers for which (3.1) holds. \square

The proof of Theorem 3.1, given in section 3.3, relies on first-order necessary conditions for a family of weak minima for auxiliary optimal control problems, namely, the *partially relaxed problems*, presented in section 3.2. These problems are defined using a *Castaing representation* of the set-valued mapping U , introduced at the beginning of section 3.2. Second-order necessary conditions in Pontryagin form in section 4.1 will be derived from a variant of the partially relaxed problems, the reduced partially relaxed problems. Thus section 3.2 is central. First- and second-order necessary conditions for a weak minimum are recalled, with some original results, in Appendix A.

3.2. Partial relaxation. In this section, (\bar{u}, \bar{y}) is a given Pontryagin minimum for problem (P) , and Assumptions 1–2 hold.

3.2.1. Castaing representation. See [10, 11, 30] for a general presentation of set-valued mappings and measurable selection theorems.

DEFINITION 3.3. Let $V : [0, T] \rightrightarrows \mathbb{R}^m$ be a set-valued mapping. We say that a sequence $(v^k)_{k \in \mathbb{N}}$, $v^k \in U$, is a Castaing representation of V iff $\{v_t^k\}_{k \in \mathbb{N}}$ is a dense subset of $V(t)$ for a.a. t .

LEMMA 3.4. There exists a Castaing representation $(u^k)_{k \in \mathbb{N}}$ of the set-valued mapping U defined by (2.20) such that, for all k , there exists $\gamma_k > 0$ such that

$$(3.2) \quad c(t, u_t^k, \bar{y}_t) \leq -\gamma_k \quad \text{for a.a. } t.$$

Proof. For $l \in \mathbb{N}$, $l \geq 1$, we consider the set-valued mapping U_l defined by

$$(3.3) \quad U_l(t) := \left\{ u \in \mathbb{R}^n : c(t, u, \bar{y}_t) \leq -\frac{1}{l} \right\} \quad \text{for a.a. } t,$$

so that

$$(3.4) \quad U(t) = \text{cl}(\cup_{l \geq 1} U_l(t)) \quad \text{for a.a. } t.$$

Under Assumptions 1–2, by [10, Theorem 3.5], and for l large enough, U_l is a measurable and nonempty closed set-valued mapping. Then by [10, Theorem 5.4], it has a Castaing representation. By (3.4), the union of such Castaing representations for l large enough is a Castaing representation of U . \square

We define the following sequence (which depends on the choice of the Castaing representation) of sets of generalized Lagrange multipliers: for $N \in \mathbb{N}$, let

$$(3.5) \quad \Lambda^N(\bar{u}, \bar{y}) := \left\{ \lambda \in \Lambda_L(\bar{u}, \bar{y}) : \begin{array}{l} H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) \leq H[p_t^\lambda](t, u_t^k, \bar{y}_t) \\ \text{for all } k \leq N, \text{ for a.a. } t \end{array} \right\}.$$

Observe that

$$(3.6) \quad \Lambda_P(\bar{u}, \bar{y}) \subset \Lambda^{N+1}(\bar{u}, \bar{y}) \subset \Lambda^N(\bar{u}, \bar{y}) \subset \Lambda_L(\bar{u}, \bar{y}),$$

and by the density of the Castaing representation,

$$(3.7) \quad \Lambda_P(\bar{u}, \bar{y}) = \bigcap_{N \in \mathbb{N}} \Lambda^N(\bar{u}, \bar{y}).$$

Recall that E and $\|\cdot\|_E$ have been defined by (2.10) and (2.11).

LEMMA 3.5. Let $(\lambda^N)_{N \in \mathbb{N}}$ be a sequence in $\Lambda_L(\bar{u}, \bar{y})$ such that $\|\lambda^N\|_E = 1$ and $\lambda^N \in \Lambda^N(\bar{u}, \bar{y})$ for all N . Then the sequence has at least one nonzero weak $*$ limit point that belongs to $\Lambda_P(\bar{u}, \bar{y})$.

Proof. By Assumption 2 and [7, Theorem 3.1], the sequence is bounded in E for the usual norm, i.e., with $\|\nu\|_\infty$ instead of $\|\nu\|_1$. Then there exists $\bar{\lambda}$ such that, extracting a subsequence if necessary, $\lambda^N \rightharpoonup \bar{\lambda}$ for the weak $*$ topology. Since $N(\bar{u}, \bar{y})$ is weakly $*$ closed, $\bar{\lambda} \in N(\bar{u}, \bar{y})$. Observe now that if $\lambda \in N(\bar{u}, \bar{y})$, then

$$(3.8) \quad \|\lambda\|_E = \beta + |\Psi| + \langle \nu, 1 \rangle_1 + \langle \mu, 1 \rangle_C,$$

where $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_C$ are the dual products in $L^1(0, T; \mathbb{R}^{n_c})$ and $C([0, T]; \mathbb{R}^{n_g})$, respectively, and 1 denotes *unit functions*, whose coordinates (in appropriate numbers) are all equal to 1. Then $\|\bar{\lambda}\|_E = 1$ and therefore $\bar{\lambda} \neq 0$. Let $p^N := p^{\lambda^N}$, $N \in \mathbb{N}$, and $\bar{p} := p^{\bar{\lambda}}$. By Lemma 2.3, $dp^N \rightharpoonup d\bar{p}$ for the weak $*$ topology in $\mathcal{M}([0, T]; \mathbb{R}^{n^*})$ and $p_T^N \rightarrow \bar{p}_T$. Since

$$(3.9) \quad p_0 = p_T - \langle dp, 1 \rangle_C$$

for any $p \in \mathcal{P}$, we derive that $\bar{p}_0 = D_{y_0} \Phi[\bar{\beta}, \bar{\Psi}](\bar{y}_0, \bar{y}_T)$. Then \bar{p} is an associated costate, i.e., $\bar{\lambda} \in N_\pi(\bar{u}, \bar{y})$. Next, as a consequence of Lemma 2.3, $p^N \rightharpoonup \bar{p}$ for the weak $*$ topology in L^∞ . Then $D_u H^a[p^N, \nu^N](\cdot, \bar{u}, \bar{y}) \rightharpoonup D_u H^a[\bar{p}, \bar{\nu}](\cdot, \bar{u}, \bar{y})$ for the weak $*$ topology in L^∞ , and then $D_u H^a[\bar{p}_t, \bar{\nu}_t](t, \bar{u}_t, \bar{y}_t) = 0$ for a.a. t , i.e., $\bar{\lambda} \in \Lambda_L(\bar{u}, \bar{y})$. Similarly, for all $k \in \mathbb{N}$,

$$(3.10) \quad H[p^N](\cdot, u^k, \bar{y}) - H[p^N](\cdot, \bar{u}, \bar{y}) \rightharpoonup H[\bar{p}](\cdot, u^k, \bar{y}) - H[\bar{p}](\cdot, \bar{u}, \bar{y})$$

for the weak $*$ topology in L^∞ , and then

$$(3.11) \quad H[\bar{p}_t](t, u_t^k, \bar{y}_t) - H[\bar{p}_t](t, \bar{u}_t, \bar{y}_t) \geq 0 \quad \text{for a.a. } t,$$

i.e., $\bar{\lambda} \in \Lambda^k(\bar{u}, \bar{y})$, for all $k \in \mathbb{N}$. By (3.7), $\bar{\lambda} \in \Lambda_P(\bar{u}, \bar{y})$. \square

Since $\Lambda^N(\bar{u}, \bar{y})$, $N \in \mathbb{N}$, are cones of nonzero elements (see Remark 2.8), it is enough to show that they are nonempty for all N to prove Theorem 3.1, by Lemma 3.5. This is the purpose of the *partially relaxed problems*, presented in the next section. Indeed, we will see that they are such that their Lagrange multipliers, whose existence can easily be guaranteed, belong to $\Lambda^N(\bar{u}, \bar{y})$.

3.2.2. The partially relaxed problems. As motivated above, we introduce now a sequence of optimal control problems.

Formulation. Recall that (\bar{u}, \bar{y}) is given as a Pontryagin minimum for problem (P).

Let $N \in \mathbb{N}$. Consider the *partially relaxed state equation*

$$(3.12) \quad \dot{y}_t = \left(1 - \sum_{i=1}^N \alpha_t^i \right) f(t, u_t, y_t) + \sum_{i=1}^N \alpha_t^i f(t, u_t^i, y_t) \quad \text{for a.a. } t \in (0, T).$$

The u^i are elements of the Castaing representation given by Lemma 3.4. The controls are u and α , the state is y , with

$$(3.13) \quad u \in \mathcal{U}, \quad \alpha \in \mathcal{A}^N := L^\infty(0, T; \mathbb{R}^N), \quad y \in \mathcal{Y}.$$

The idea is to consider the problem of minimizing $\phi(y_0, y_T)$ under the same constraints as before, plus the control constraints $\alpha \geq 0$. To simplify the qualification issue, we actually introduce a *slack variable* $\theta \in \mathbb{R}$, with the intention of minimizing it, and with the following constraint on the cost function:

$$(3.14) \quad \phi(y_0, y_T) - \phi(\bar{y}_0, \bar{y}_T) \leq \theta.$$

The slack variable θ also enters into every inequality constraint:

$$(3.15) \quad -\alpha_t \leq \theta \quad \text{for a.a. } t \in (0, T),$$

$$(3.16) \quad c(t, u_t, y_t) \leq \theta \quad \text{for a.a. } t \in (0, T),$$

$$(3.17) \quad g(t, y_t) \leq \theta \quad \text{for a.a. } t \in (0, T),$$

$$(3.18) \quad \Phi^I(y_0, y_T) \leq \theta,$$

and the equality constraints remain unchanged:

$$(3.19) \quad \Phi^E(y_0, y_T) = 0.$$

The *partially relaxed problem* is

$$(P_N) \quad \min_{(u, \alpha, y, \theta) \in \mathcal{U} \times \mathcal{A}^N \times \mathcal{Y} \times \mathbb{R}} \theta \quad \text{subject to} \quad (3.12)\text{--}(3.19).$$

As for problem (P) , we call a *relaxed trajectory* any (u, α, y, θ) such that (3.12) holds. We say that a relaxed trajectory is feasible if it satisfies constraints (3.14)–(3.19), and denote by $F(P_N)$ the set of feasible relaxed trajectories.

Let $\bar{\alpha} := 0 \in \mathcal{A}^N$ and $\bar{\theta} := 0 \in \mathbb{R}$. Then $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta}) \in F(P_N)$. Under Assumption 1, for any (u, α, y^0) close to $(\bar{u}, \bar{\alpha}, \bar{y})$ in $\mathcal{U} \times \mathcal{A}^N \times \mathbb{R}^n$, there exists a unique $y \in \mathcal{Y}$ such that (3.12) holds and $y_0 = y^0$; we denote it by $y[u, \alpha, y^0]$ and consider the mapping

$$(3.20) \quad \Gamma_N : (u, \alpha, y^0) \mapsto y[u, \alpha, y^0].$$

Remark 3.6. Robinson’s constraint qualification holds at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ iff the equality constraints are qualified, i.e., iff the derivative of

$$(3.21) \quad (u, \alpha, y^0) \in \mathcal{U} \times \mathcal{A}^N \times \mathbb{R}^n \mapsto \Phi^E(y^0, \Gamma_N(u, \alpha, y^0)_T) \in \mathbb{R}^{n_{\Phi^E}}$$

at $(\bar{u}, \bar{\alpha}, \bar{y}_0)$ is onto. We say that problem (P_N) is *qualified* at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ if this is the case. See [9, section 2.3.4] for the definition and characterizations of Robinson’s constraint qualification.

Existence of a minimum. A key result is the following.

THEOREM 3.7. *Let Assumptions 1–2 hold and let problem (P_N) be qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$. Then $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ is a weak minimum for this problem.*

Theorem 3.7 is a corollary of the following proposition, which can be deduced from classical relaxation theorems, such as [14, Theorem 3]. Note that a complete proof can be found in [3, section A.2].

PROPOSITION 3.8. *Under the assumptions of Theorem 3.7, there exists $M > 0$ such that, for any $(\hat{u}, \hat{\alpha}, \hat{y}, \hat{\theta}) \in F(P_N)$ in an L^∞ neighborhood of $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ and with $\hat{\theta} < 0$, for any $\varepsilon > 0$, there exists $(\tilde{u}, \tilde{y}) \in F(P)$ such that*

$$(3.22) \quad \|\tilde{u} - \hat{u}\|_1 \leq M \|\hat{\alpha}\|_\infty \quad \text{and} \quad \|\tilde{y} - \hat{y}\|_\infty \leq \varepsilon.$$

Proof of Theorem 3.7. Suppose that $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ is not a weak minimum for problem (P_N) . Then there exists $(\hat{u}, \hat{\alpha}, \hat{y}, \hat{\theta}) \in F(P_N)$ arbitrarily close to $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ in the L^∞ norm and with $\hat{\theta} < 0$. Let $\varepsilon > 0$ be such that

$$(3.23) \quad \|y - \hat{y}\|_\infty \leq \varepsilon \Rightarrow \phi(y_0, y_T) < \phi(\bar{y}_0, \bar{y}_T).$$

By Proposition 3.8, we get $(\bar{u}, \bar{y}) \in F(P)$ such that $\phi(\bar{y}_0, \bar{y}_T) < \phi(\bar{y}_0, \bar{y}_T)$ and

$$(3.24) \quad \|\bar{u} - \bar{u}\|_1 + \|\bar{y} - \bar{y}\|_\infty \leq M \|\hat{\alpha}\|_\infty + T\|\hat{u} - \bar{u}\|_\infty + \varepsilon + \|\hat{y} - \bar{y}\|_\infty.$$

Observe that the right-hand side of (3.24) can be chosen as small as needed. Thus we get a contradiction with the Pontryagin optimality of (\bar{u}, \bar{y}) . \square

Optimality conditions. Problem (P_N) can be seen as an optimization problem over $(u, \alpha, y^0, \theta) \in \mathcal{U} \times \mathcal{A}^N \times \mathbb{R}^n \times \mathbb{R}$, via the mapping Γ_N defined by (3.20). Then we can define the set $\Lambda(P_N)$ of Lagrange multipliers at $(\bar{u}, \bar{\alpha}, \bar{y}_0, \bar{\theta})$ as in Appendix A:

$$(3.25) \quad \Lambda(P_N) := \{(\lambda, \gamma) \in N(\bar{u}, \bar{y}) \times L^\infty(0, T; \mathbb{R}_+^{N*}) : DL_N[\lambda, \gamma](\bar{u}, \bar{\alpha}, \bar{y}_0, \bar{\theta}) = 0\},$$

where L_N is defined, for $\lambda = (\beta, \Psi, \nu, \mu)$, $\Psi = (\Psi^E, \Psi^I)$, $y = \Gamma_N(u, \alpha, y^0)$, by

$$(3.26) \quad \begin{aligned} &L_N[\lambda, \gamma](u, \alpha, y^0, \theta) \\ &:= \theta + \beta(\phi(y_0, y_T) - \phi(\bar{y}_0, \bar{y}_T) - \theta) \\ &+ \Psi^E \Phi^E(y_0, y_T) + \Psi^I(\Phi^I(y_0, y_T) - \theta 1) \\ &+ \int_{[0, T]} [\nu_t(c(t, u_t, y_t) - \theta 1) dt + d\mu_t(g(t, y_t) - \theta 1) - \gamma_t(\alpha_t + \theta 1) dt]. \end{aligned}$$

In (3.26), 1 represents already defined unit functions. We have the following first-order necessary conditions.

LEMMA 3.9. *Let problem (P_N) be qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$. Then $\Lambda(P_N)$ is nonempty, convex, and weakly $*$ compact.*

Proof. We apply Theorem A.7 to $(\bar{u}, \bar{\alpha}, \bar{y}_0, \bar{\theta})$, a locally optimal solution of (P_N) by Theorem 3.7. Let $\bar{v} \in \mathcal{U}$ be given by the inward condition for the mixed constraints in problem (P) (Assumption 2) and let $\bar{\omega} := 1 \in \mathcal{A}^N$. Then $(\bar{v}, \bar{\omega})$ satisfies the inward condition for the mixed constraints in problem (P_N) . The other assumptions being also satisfied by Assumption 1 and Remark 3.6, the conclusion follows. \square

3.3. Proof of Theorem 3.1. As explained at the end of section 3.2.1, it is enough by Lemma 3.5 to prove that $\Lambda^N(\bar{u}, \bar{y}) \neq \emptyset$ for all N . To do so, we use the partially relaxed problems (P_N) as follows.

LEMMA 3.10. *Let $(\lambda, \gamma) \in \Lambda(P_N)$. Then $\lambda \in \Lambda^N(\bar{u}, \bar{y})$.*

Proof. Let (u, α, y, θ) be a relaxed trajectory and $(\lambda, \gamma) \in E \times L^\infty(0, T; \mathbb{R}^{N*})$ with $\lambda = (\beta, \Psi, \nu, \mu)$ and $\Psi = (\Psi^E, \Psi^I)$. Adding to L_N

$$(3.27) \quad 0 = \int_0^T p_t \left(\left(1 - \sum_{i=1}^N \alpha_t^i \right) f(t, u_t, y_t) + \sum_{i=1}^N \alpha_t^i f(t, u_t^i, y_t) - \dot{y}_t \right) dt,$$

and integrating by parts we have, for any $p \in \mathcal{P}$,

$$(3.28) \quad \begin{aligned} &L_N[\lambda, \gamma](u, \alpha, y_0, \theta) \\ &= \theta (1 - \beta - \langle \Psi^I, 1 \rangle - \langle \nu, 1 \rangle_1 - \langle \mu, 1 \rangle_C - \langle \gamma, 1 \rangle_1) \\ &+ \int_0^T \left(H^a[p_t, \nu_t](t, u_t, y_t) + \sum_{i=1}^N \alpha_t^i (H[p_t^i](t, u_t^i, y_t) - H[p_t](t, u_t, y_t) - \gamma_t^i) \right) dt \\ &+ \int_{[0, T]} (d\mu_t g(t, y_t) + dp_t y_t) + \Phi[\beta, \Psi](y_0, y_T) - p_T y_T + p_0 y_0 - \beta \phi(\bar{y}_0, \bar{y}_T). \end{aligned}$$

Let $(\lambda, \gamma) \in \Lambda(P_N)$. Using the expression (3.28) of L_N , we get

$$(3.29) \quad D_{y_0} \Phi[\beta, \Psi](\bar{y}_0, \bar{y}_T) + p_0^\lambda = 0,$$

$$(3.30) \quad D_u H^\alpha[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t) = 0 \quad \text{for a.a. } t,$$

$$(3.31) \quad H[p_t^\lambda](t, u_t^i, \bar{y}_t) - H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) = \gamma_t^i \quad \text{for a.a. } t, \quad 1 \leq i \leq N,$$

$$(3.32) \quad \beta + \langle \Psi^I, 1 \rangle + \langle \nu, 1 \rangle_1 + \langle \mu, 1 \rangle_C + \langle \gamma, 1 \rangle_1 = 1.$$

Suppose that $\lambda = 0$. Then $p^\lambda = 0$ and by (3.31), $\gamma = 0$; we get a contradiction with (3.32). Then $\lambda \neq 0$ and $\lambda \in N_\pi(\bar{u}, \bar{y})$ by (3.29). Finally, $\lambda \in \Lambda_L(\bar{u}, \bar{y})$ by (3.30), and $\lambda \in \Lambda^N(\bar{u}, \bar{y})$ by (3.31) since $\gamma \in L^\infty(0, T; \mathbb{R}_+^{N*})$. \square

We need one more lemma.

LEMMA 3.11. *Let problem (P_N) be not qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$. Then there exists $\lambda \in \Lambda^N(\bar{u}, \bar{y})$ such that $-\lambda \in \Lambda^N(\bar{u}, \bar{y})$ too, and for all $k \leq N$,*

$$(3.33) \quad H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) = H[p_t^\lambda](t, u_t^i, \bar{y}_t) \quad \text{for a.a. } t.$$

Proof. Recall that Γ_N has been defined by (3.20). By Remark 3.6, there exists $\Psi^E \neq 0$ such that

$$(3.34) \quad \Psi^E D\Phi^E(\bar{y}_0, \bar{y}_T) D\Gamma_N(\bar{u}, \bar{\alpha}, \bar{y}_0) = 0.$$

Let $\Psi = (\Psi^E, 0)$ and $\lambda := (0, \Psi, 0, 0)$, so that $D_{(u, \alpha, y^0)} L_N[\lambda, 0](\bar{u}, \bar{\alpha}, \bar{y}_0, \bar{\theta}) = 0$ by (3.26). By (3.28), we get

$$(3.35) \quad D_{y_0} \Phi[0, (\Psi^E, 0)](\bar{y}_0, \bar{y}_T) + p_0^\lambda = 0,$$

$$(3.36) \quad D_u H^\alpha[p_t^\lambda, 0](t, \bar{u}_t, \bar{y}_t) = 0 \quad \text{for a.a. } t,$$

$$(3.37) \quad H[p_t^\lambda](t, u_t^i, \bar{y}_t) - H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) = 0 \quad \text{for a.a. } t, \quad 1 \leq i \leq N.$$

Then $-\lambda, \lambda \in \Lambda^N(\bar{u}, \bar{y})$ and (3.33) holds. \square

We can now conclude the following.

Proof of Theorem 3.1. We need $\Lambda^N(\bar{u}, \bar{y}) \neq \emptyset$ for all N . If problem (P_N) is qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$, then $\Lambda^N(\bar{u}, \bar{y}) \neq \emptyset$ by Lemmas 3.9 and 3.10. If problem (P_N) is not qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$, then $\Lambda^N(\bar{u}, \bar{y}) \neq \emptyset$ by Lemma 3.11. \square

Actually, we have the following alternative.

COROLLARY 3.12. *The partially relaxed problems (P_N) are either qualified for all N large enough, if $\Lambda_P^D(\bar{u}, \bar{y}) = \emptyset$, or never qualified, and then $\Lambda_P^D(\bar{u}, \bar{y}) \neq \emptyset$.*

Proof. If the problems (P_N) are never qualified, then we get a sequence of multipliers as in the proof of Lemma 3.11. By the proof of Lemma 3.5, its limit points belong to $\Lambda_P^D(\bar{u}, \bar{y})$. \square

See Appendix B for a qualification condition ensuring the nonsingularity of the generalized Pontryagin multipliers.

4. Second-order conditions in Pontryagin form.

4.1. Statement. The second-order necessary conditions presented in this section involve Pontryagin multipliers only. They rely again on the partially relaxed problems, introduced in section 3.2. These problems are actually modified into *reduced* partially relaxed problems, which satisfy an *extended polyhedricity condition* [9, section 3.2.3]. The idea is to get our second-order necessary conditions on a large cone by density of the so-called *strict radial critical cone*, so that we do not have to compute the *envelope-like effect* (Kawasaki [23]).

The main result of this section is Theorem 4.9. It is stated after some new definitions and assumptions, and proved in section 4.2.

4.1.1. Definitions and assumptions. For second-order optimality conditions, we need a stronger regularity assumption than Assumption 1. Namely, we make the following assumption in what follows.

Assumption 3. The mappings f and g are C^∞ , c is uniformly quasi- C^2 , Φ and ϕ are C^2 .

Remark 4.1. If there is no pure state constraint in problem (P) (i.e., no mapping g), we will see that it is enough to assume that f is uniformly quasi- C^2 .

For $s \in [1, \infty]$, let

$$(4.1) \quad \mathcal{V}_s := L^s(0, T; \mathbb{R}^m), \quad \mathcal{Z}_s := W^{1,s}(0, T; \mathbb{R}^n).$$

Let (\bar{u}, \bar{y}) be a trajectory for problem (P) . Given $v \in \mathcal{V}_s$, $s \in [1, \infty]$, we consider the linearized state equation in \mathcal{Z}_s

$$(4.2) \quad \dot{z}_t = Df(t, \bar{u}_t, \bar{y}_t)(v_t, z_t) \quad \text{for a.a. } t \in (0, T).$$

We call a *linearized trajectory* any $(v, z) \in \mathcal{V}_s \times \mathcal{Z}_s$ such that (4.2) holds. For any $(v, z^0) \in \mathcal{V}_s \times \mathbb{R}^n$, there exists a unique $z \in \mathcal{Z}_s$ such that (4.2) holds and $z_0 = z^0$; we denote it by $z = z[v, z^0]$.

For $1 \leq i \leq n_g$, we define $g_i^{(j)} : [0, T] \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$, $j \in \mathbb{N}$, recursively by

$$(4.3) \quad g_i^{(j+1)}(t, u, y) := D_t g_i^{(j)}(t, u, y) + D_y g_i^{(j)}(t, u, y)f(t, u, y), \quad g_i^{(0)} := g_i.$$

DEFINITION 4.2. The order of a state constraint g_i is $q_i \in \mathbb{N}$ such that

$$(4.4) \quad D_u g_i^{(j)} \equiv 0 \quad \text{for } 0 \leq j \leq q_i - 1, \quad D_u g_i^{(q_i)} \neq 0.$$

Remark 4.3. If g_i is of order q_i , then $t \mapsto g_i(t, \bar{y}_t) \in W^{q_i, \infty}(0, T)$ for any trajectory (\bar{u}, \bar{y}) , and

$$(4.5) \quad \frac{d^j}{dt^j} g_i(t, \bar{y}_t) = g_i^{(j)}(t, \bar{y}_t) \quad \text{for } 0 \leq j \leq q_i - 1,$$

$$(4.6) \quad \frac{d^{q_i}}{dt^{q_i}} g_i(t, \bar{y}_t) = g_i^{(q_i)}(t, \bar{u}_t, \bar{y}_t).$$

We have the same regularity along linearized trajectories; the proof of the next lemma is classical; see, for instance, [5, Lemma 9].

LEMMA 4.4. Let (\bar{u}, \bar{y}) be a trajectory and $(v, z) \in \mathcal{V}_s \times \mathcal{Z}_s$ be a linearized trajectory, $s \in [1, \infty]$. Let the constraint g_i be of order q_i . Then

$$(4.7) \quad t \mapsto Dg_i(t, \bar{y}_t)z_t \in W^{q_i, s}(0, T),$$

and

$$(4.8) \quad \frac{d^j}{dt^j} Dg_i(t, \bar{y}_t)z_t = Dg_i^{(j)}(t, \bar{y}_t)z_t \quad \text{for } 0 \leq j \leq q_i - 1,$$

$$(4.9) \quad \frac{d^{q_i}}{dt^{q_i}} Dg_i(t, \bar{y}_t)z_t = Dg_i^{(q_i)}(t, \bar{u}_t, \bar{y}_t)(v_t, z_t).$$

DEFINITION 4.5. Let $(\bar{u}, \bar{y}) \in F(P)$. We say that $\tau \in [0, T]$ is a touchpoint for the constraint g_i iff it is a contact point for g_i , i.e., $g_i(\tau, \bar{y}_\tau) = 0$, isolated in

$\{t : g_i(t, \bar{y}_t) = 0\}$. We say that a touchpoint τ for g_i is reducible iff $\tau \in (0, T)$, $\frac{d^2}{dt^2}g_i(t, \bar{y}_t)$ is defined for t close to τ , continuous at τ , and

$$(4.10) \quad \frac{d^2}{dt^2}g_i(t, \bar{y}_t)|_{t=\tau} < 0.$$

Remark 4.6. If g_i is of order at least 2, then by Remark 4.3 a touchpoint τ for g_i is reducible iff $t \mapsto g_i^{(2)}(t, \bar{u}_t, \bar{y}_t)$ is continuous at τ and $g_i^{(2)}(\tau, \bar{u}_\tau, \bar{y}_\tau) < 0$. The continuity holds if \bar{u} is continuous at τ or if g_i is of order at least 3.

Let $(\bar{u}, \bar{y}) \in F(P)$. For $1 \leq i \leq n_g$, let

$$(4.11) \quad \mathcal{T}_{g,i} := \begin{cases} \emptyset & \text{if } g_i \text{ is of order 1,} \\ \{\text{touchpoints for } g_i\} & \text{if } g_i \text{ is of order at least 2,} \end{cases}$$

$$(4.12) \quad \Delta_{g,i}^0 := \{t \in [0, T] : g_i(t, \bar{y}_t) = 0\} \setminus \mathcal{T}_{g,i},$$

$$(4.13) \quad \Delta_{g,i}^\varepsilon := \{t \in [0, T] : \text{dist}(t, \Delta_{g,i}^0) \leq \varepsilon\},$$

and for $1 \leq i \leq n_c$, let

$$(4.14) \quad \Delta_{c,i}^\delta := \{t \in [0, T] : c_i(t, \bar{u}_t, \bar{y}_t) \geq -\delta\}.$$

We will need the following two extra assumptions.

Assumption 4. For $1 \leq i \leq n_g$, the set $\mathcal{T}_{g,i}$ is finite and contains only reducible touchpoints, $\Delta_{g,i}^0$ has finitely many connected components, and g_i is of finite order q_i .

Assumption 5. There exist $\delta', \varepsilon' > 0$ such that the linear mapping from $\mathcal{V}_2 \times \mathbb{R}^n$ to $\prod_{i=1}^{n_c} L^2(\Delta_{c,i}^{\delta'}) \times \prod_{i=1}^{n_g} W^{q_i, 2}(\Delta_{g,i}^{\varepsilon'})$ defined by

$$(4.15) \quad (v, z^0) \mapsto \left(\begin{array}{l} (Dc_i(\cdot, \bar{u}, \bar{y})(v, z[v, z^0])|_{\Delta_{c,i}^{\delta'}})_{1 \leq i \leq n_c} \\ (Dg_i(\cdot, \bar{y})z[v, z^0]|_{\Delta_{g,i}^{\varepsilon'}})_{1 \leq i \leq n_g} \end{array} \right) \text{ is onto.}$$

Remark 4.7.

- (i) Assumption 4 discards the case of optimal trajectories with infinitely many touchpoints that occurs for state constraints of order greater than 2 with boundary arcs; see Robbins [28]. It still covers many cases, as in [20, Example 4.6].
- (ii) Under a strong qualification condition (of linear independence type), Assumption 5 is satisfied; see, e.g., [6, Lemma 2.3] or [2, Lemma 4.5], and again [20, Example 4.6].
- (iii) It is not easy to give a simple example of a problem having a discontinuous optimal control satisfying Assumptions 4 and 5.

4.1.2. Main result. Let $(\bar{u}, \bar{y}) \in F(P)$. We define the *critical cone* in L^2

$$(4.16) \quad C_2(\bar{u}, \bar{y}) := \left\{ \begin{array}{l} (v, z) \in \mathcal{V}_2 \times \mathcal{Z}_2 : z = z[v, z_0] \\ D\phi(\bar{y}_0, \bar{y}_T)(z_0, z_T) \leq 0 \\ D\Phi(\bar{y}_0, \bar{y}_T)(z_0, z_T) \in T_{K_\Phi}(\Phi(\bar{y}_0, \bar{y}_T)) \\ Dc(\cdot, \bar{u}, \bar{y})(v, z) \in T_{K_c}(c(\cdot, \bar{u}, \bar{y})) \\ Dg(\cdot, \bar{y})z \in T_{K_g}(g(\cdot, \bar{y})) \end{array} \right\}$$

and the *strict critical cone* in L^2

$$(4.17) \quad C_2^S(\bar{u}, \bar{y}) := \left\{ \begin{array}{l} (v, z) \in C_2(\bar{u}, \bar{y}) : \\ Dc_i(t, \bar{u}_t, \bar{y}_t)(v_t, z_t) = 0, \quad t \in \Delta_{c,i}^0, \quad 1 \leq i \leq n_c \\ Dg_i(t, \bar{y}_t)z_t = 0, \quad t \in \Delta_{g,i}^0, \quad 1 \leq i \leq n_g \end{array} \right\}.$$

Remark 4.8.

1. See [9, Examples 2.63 and 2.64] for the description of T_{K_g} and T_{K_c} , respectively.
2. Since by Assumption 4 there are finitely many touchpoints for constraints of order at least 2, $C_2^S(\bar{u}, \bar{y})$ is defined by equality constraints and a finite number of inequality constraints, i.e., the cone $C_2^S(\bar{u}, \bar{y})$ is a polyhedron of a closed subspace.
3. The strict critical cone $C_2^S(\bar{u}, \bar{y})$ is a subset of the critical cone $C_2(\bar{u}, \bar{y})$. But if there exists $\lambda = (\bar{\beta}, \bar{\Psi}, \bar{\nu}, \bar{\mu}) \in \Lambda_L(\bar{u}, \bar{y})$ such that

$$(4.18) \quad \bar{\nu}_i(t) > 0 \quad \text{for a.a. } t \in \Delta_{c,i}^0, \quad 1 \leq i \leq n_c,$$

$$(4.19) \quad \Delta_{g,i}^0 = \text{supp}(\bar{\mu}_i), \quad 1 \leq i \leq n_g,$$

then $C_2^S(\bar{u}, \bar{y}) = C_2(\bar{u}, \bar{y})$ (see [9, Proposition 3.10]).

For any $\lambda = (\beta, \Psi, \nu, \mu) \in E$, we define a quadratic form $\Omega[\lambda]: \mathcal{V}_2 \times \mathcal{Z}_2 \rightarrow \mathbb{R}$ by

$$(4.20) \quad \begin{aligned} \Omega[\lambda](v, z) := & \int_0^T D^2 H^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t)(v_t, z_t)^2 dt + D^2 \Phi[\beta, \Psi](\bar{y}_0, \bar{y}_T)(z_0, z_T)^2 \\ & + \int_{[0,T]} d\mu_t D^2 g(t, \bar{y}_t)(z_t)^2 - \sum_{\substack{\tau \in T_{g,i} \\ 1 \leq i \leq n_g}} \mu_i(\tau) \frac{\left(Dg_i^{(1)}(\tau, \bar{y}_\tau)z_\tau \right)^2}{g_i^{(2)}(\tau, \bar{u}_\tau, \bar{y}_\tau)}. \end{aligned}$$

We can now state our main result, that will be proved in the next section.

THEOREM 4.9. *Let (\bar{u}, \bar{y}) be a Pontryagin minimum for problem (P) and let Assumptions 2–5 hold. Then for any $(v, z) \in C_2^S(\bar{u}, \bar{y})$, there exists $\lambda \in \Lambda_P(\bar{u}, \bar{y})$ such that*

$$(4.21) \quad \Omega[\lambda](v, z) \geq 0.$$

Remark 4.10. If $\Lambda_P^D(\bar{u}, \bar{y}) \neq \emptyset$ and $\lambda \in \Lambda_P^D(\bar{u}, \bar{y})$, then $-\lambda \in \Lambda_P^D(\bar{u}, \bar{y})$ too. Since $\Omega[-\lambda](v, z) = -\Omega[\lambda](v, z)$ for any $(v, z) \in \mathcal{V}_2 \times \mathcal{Z}_2$, Theorem 4.9 is then pointless. See Corollary 3.12 about the emptiness of $\Lambda_P^D(\bar{u}, \bar{y})$.

4.2. Proof of Theorem 4.9. In this section, (\bar{u}, \bar{y}) is a given Pontryagin minimum for problem (P), and Assumptions 2–5 hold.

4.2.1. Reduction and partial relaxation. The reduction approach [5, section 5] consists in reformulating the state constraint in the neighborhood of a touchpoint, using its reducibility (Definition 4.5). We apply this approach to the partially relaxed problems (P_N) in order to involve Pontryagin multipliers (see Lemmas 3.5 and 3.10).

Let $N \in \mathbb{N}$. Recall that Γ_N has been defined by (3.20).

Remark 4.11. The result of Remark 4.3 still holds for relaxed trajectories:

$$(4.22) \quad t \mapsto g_i(t, y_t) \in W^{q_i, \infty}(0, T) \quad \text{for any } y = \Gamma_N(u, \alpha, y^0).$$

Let $\tau \in \mathcal{T}_{g,i}$. We define $\Theta_{i,\tau}^{\varepsilon,N} : \mathcal{U} \times \mathcal{A}^N \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$(4.23) \quad \Theta_{i,\tau}^{\varepsilon,N}(u, \alpha, y^0) := \max \{g_i(t, y_t) : y = \Gamma_N(u, \alpha, y^0), t \in [\tau - \varepsilon, \tau + \varepsilon] \cap [0, T]\}.$$

Let $\bar{\Gamma}'_N := D\Gamma_N(\bar{u}, \bar{\alpha}, \bar{y}_0)$ and $\bar{\Gamma}''_N := D^2\Gamma_N(\bar{u}, \bar{\alpha}, \bar{y}_0)$.

Remark 4.12. Let $\bar{\omega} := 0 \in \mathcal{A}^N$. For any $(v, z^0) \in \mathcal{V}_s \times \mathbb{R}^n$, $s \in [1, \infty]$, we have

$$(4.24) \quad \bar{\Gamma}'_N(v, \bar{\omega}, z^0) = z[v, z^0].$$

LEMMA 4.13. *There exists $\varepsilon > 0$ independent of N such that for any $\tau \in \mathcal{T}_{g,i}$, $\Theta_{i,\tau}^{\varepsilon,N}$ is C^1 in a neighborhood of $(\bar{u}, \bar{\alpha}, \bar{y}_0)$ and twice Fréchet differentiable at $(\bar{u}, \bar{\alpha}, \bar{y}_0)$, with first and second derivatives given by*

$$(4.25) \quad D\Theta_{i,\tau}^{\varepsilon,N}(\bar{u}, \bar{\alpha}, \bar{y}_0)(v, \omega, z^0) = Dg_i(\tau, \bar{y}_\tau)\bar{\Gamma}'_N(v, \omega, z^0)_\tau$$

for any $(v, \omega, z^0) \in \mathcal{V}_1 \times L^1(0, T; \mathbb{R}^N) \times \mathbb{R}^n$, and

$$(4.26) \quad \begin{aligned} &D^2\Theta_{i,\tau}^{\varepsilon,N}(\bar{u}, \bar{\alpha}, \bar{y}_0)(v, \omega, z^0)^2 \\ &= D^2g_i(\tau, \bar{y}_\tau)(\bar{\Gamma}'_N(v, \omega, z^0)_\tau)^2 \\ &+ Dg_i(\tau, \bar{y}_\tau)\bar{\Gamma}''_N(v, \omega, z^0)_\tau^2 - \frac{(\frac{d}{dt}Dg_i(\cdot, \bar{y})\bar{\Gamma}'_N(v, \omega, z^0)|_\tau)^2}{\frac{d^2}{dt^2}g_i(\cdot, \bar{y})|_\tau} \end{aligned}$$

for any $(v, \omega, z^0) \in \mathcal{V}_2 \times L^2(0, T; \mathbb{R}^N) \times \mathbb{R}^n$.

Proof. Combine [5, Lemma 23] with Remark 4.11 and Assumption 4. □

The reduced partially relaxed problems. The formulation is the same as for problems (P_N) , except that (i) we localize the mixed constraints c and the state constraints g on the domains given by Assumption 5, (ii) we replace the state constraints of order at least 2 around their touchpoints with the mappings $\Theta_{i,\tau}^{\varepsilon,N}$. Without loss of generality we assume that ε' given by Assumption 5 is smaller than ε given by Lemma 4.13; δ' is also given by Assumption 5.

Let $N \in \mathbb{N}$. Recall that in section 3.2 the partially relaxed problem was

$$(P_N) \quad \min_{(u, \alpha, y, \theta) \in \mathcal{U} \times \mathcal{A}^N \times \mathcal{Y} \times \mathbb{R}} \theta \quad \text{subject to} \quad (3.12)\text{--}(3.19).$$

We consider the following new constraints:

$$(4.27) \quad c_i(t, u_t, y_t) \leq \theta \quad \text{for a.a. } t \in \Delta_{c,i}^{\delta'}, \quad 1 \leq i \leq n_c,$$

$$(4.28) \quad g_i(t, y_t) \leq \theta \quad \text{for a.a. } t \in \Delta_{g,i}^{\varepsilon'}, \quad 1 \leq i \leq n_g,$$

$$(4.29) \quad \Theta_{i,\tau}^{\varepsilon,N}(u, \alpha, y_0) \leq \theta \quad \text{for all } \tau \in \mathcal{T}_{g,i}, \quad 1 \leq i \leq n_g.$$

The *reduced partially relaxed problem* is then

$$(P_N^R) \quad \min_{(u, \alpha, y, \theta) \in \mathcal{U} \times \mathcal{A}^N \times \mathcal{Y} \times \mathbb{R}} \theta \quad \text{subject to} \quad (3.12)\text{--}(3.15), (3.18)\text{--}(3.19), (4.27)\text{--}(4.29).$$

As before, we denote by $F(P_N^R)$ the set of feasible relaxed trajectories.

Remark 4.14.

1. We have $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta}) \in F(P_N^R)$ and, in a neighborhood of $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$, $(u, \alpha, y, \theta) \in F(P_N^R)$ iff $(u, \alpha, y, \theta) \in F(P_N)$. In particular, $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ is a weak minimum for problem (P_N^R) iff it is a weak minimum for problem (P_N) .
2. Problem (P_N^R) is qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ iff problem (P_N) is qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ (see Remark 3.6).

Optimality conditions. Again, problem (P_N^R) can be seen as an optimization problem over (u, α, y^0, θ) , via the mapping Γ_N . We denote its Lagrangian by L_N^R , its set of Lagrange multipliers at $(\bar{u}, \bar{\alpha}, \bar{y}_0, \bar{\theta})$ by $\Lambda(P_N^R)$, and its set of quasi-radial critical directions in L^2 by $C_2^{QR}(P_N^R)$, as defined in Appendix A.

Remark 4.15. By Lemma 4.13, we can identify $\Lambda(P_N^R)$ and $\Lambda(P_N)$ by identifying the scalar components of a multiplier associated with the constraints (4.29) and Dirac measures. See also [5, Lemma 26] or [2, Lemma 3.4].

We have the following second-order necessary conditions.

LEMMA 4.16. *Let problem (P_N^R) be qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$. Then for any $(v, \omega, z^0, \vartheta) \in \text{cl}(C_2^{QR}(P_N^R))$, there exists $(\lambda, \gamma) \in \Lambda(P_N^R)$ such that*

$$(4.30) \quad D^2L_N^R[\lambda, \gamma](v, \omega, z^0, \vartheta)^2 \geq 0.$$

Here, cl denotes the L^2 closure.

Proof. We apply Theorem A.8 to $(\bar{u}, \bar{\alpha}, \bar{y}_0, \bar{\theta})$, a locally optimal solution of (P_N^R) by Theorem 3.7, and Remark 4.14. The various mappings have the required regularity by Assumption 3 and Lemma 4.13. Robinson’s constraint qualification and the inward condition for the mixed constraints hold as in the proof of Lemma 3.9. The conclusion follows. \square

4.2.2. Proof of the theorem. Let $(\bar{v}, \bar{z}) \in C_2^S(\bar{u}, \bar{y})$. By Lemma 3.5 and since $\lambda \mapsto \Omega[\lambda](\bar{v}, \bar{z})$ is linear continuous, it is enough to show that for all N , there exists $\lambda^N \in \Lambda^N(\bar{u}, \bar{y})$ such that

$$(4.31) \quad \Omega[\lambda^N](\bar{v}, \bar{z}) \geq 0.$$

Let $(\bar{\omega}, \bar{\vartheta}) := (0, 0) \in \mathcal{A}^N \times \mathbb{R}$. The link with the reduced partially relaxed problems (P_N^R) is as follows.

LEMMA 4.17. *Let $(\lambda, \gamma) \in \Lambda(P_N^R)$. Then $\lambda \in \Lambda^N(\bar{u}, \bar{y})$ and*

$$(4.32) \quad D^2L_N^R[\lambda, \gamma](\bar{v}, \bar{\omega}, \bar{z}_0, \bar{\vartheta})^2 = \Omega[\lambda](\bar{v}, \bar{z}).$$

Proof. The first part of the result is known by Lemma 3.10 and Remark 4.15. For the second part, we write L_N^R using H^a and H , as in the expression (3.28) of L_N , and we compute its second derivative. The result follows by Lemma 4.13 and Remark 4.12. See also [5, Lemma 26] or [2, Lemma 3.5]. \square

We also need the following density result, that will be proved in section 4.2.3.

LEMMA 4.18. *The direction $(\bar{v}, \bar{\omega}, \bar{z}_0, \bar{\vartheta})$ belongs to $\text{cl}(C_2^{QR}(P_N^R))$, the closure of the set of quasi-radial critical directions in L^2 .*

We can now conclude the following.

Proof of Theorem 4.9. We need $\lambda^N \in \Lambda^N(\bar{u}, \bar{y})$ such that (4.31) holds for all N . If problem (P_N^R) is qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$, then we get λ^N as needed by Lemmas 4.16, 4.17, and 4.18. If problem (P_N^R) is not qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$, then we get λ such that $-\lambda, \lambda \in \Lambda^N(\bar{u}, \bar{y})$ by Remark 4.14.2 and Lemma 3.11. Since $\lambda \mapsto \Omega[\lambda](\bar{v}, \bar{z})$ is linear, (4.31) holds for $\lambda^N = \pm\lambda$. \square

4.2.3. A density result. In this section we prove Lemma 4.18. Recall that δ' is given by Assumption 5. We define the *strict radial critical cone* in L^2

(4.33)

$$C_2^R(\bar{u}, \bar{y}) := \left\{ \begin{array}{l} (v, z) \in C_2(\bar{u}, \bar{y}) : \\ \exists \delta > 0 : Dc_i(t, \bar{u}_t, \bar{y}_t)(v_t, z_t) = 0, \quad t \in \Delta_{c,i}^\delta, \quad 1 \leq i \leq n_c \\ \exists M > 0 : |Dc_i(t, \bar{u}_t, \bar{y}_t)(v_t, z_t)| \leq M, \quad t \in \Delta_{c,i}^{\delta'}, \quad 1 \leq i \leq n_c \\ \exists \varepsilon > 0 : Dg_i(t, \bar{y}_t)z_t = 0, \quad t \in \Delta_{g,i}^\varepsilon, \quad 1 \leq i \leq n_g \end{array} \right\}.$$

PROPOSITION 4.19. *The strict radial critical cone $C_2^R(\bar{u}, \bar{y})$ is a dense subset of the strict critical cone $C_2^S(\bar{u}, \bar{y})$.*

Proof. Recall that by definition, touchpoints for g_i are included in $\Delta_{g,i}^\varepsilon$, $\varepsilon \geq 0$, iff g_i is of order 1.

(a) Let $W^{(q),2}(0, T) := \prod_{i=1}^{n_g} W^{q_i,2}(0, T)$. We claim that the subspace

$$(4.34) \quad \left\{ \begin{array}{l} (\phi, \psi) \in L^\infty(0, T; \mathbb{R}^{n_c}) \times W^{(q),2}(0, T) : \\ \exists \delta > 0 : \phi_{i,t} = 0, \quad t \in \Delta_{c,i}^\delta, \quad 1 \leq i \leq n_c \\ \exists \varepsilon > 0 : \psi_{i,t} = 0, \quad t \in \Delta_{g,i}^\varepsilon, \quad 1 \leq i \leq n_g \end{array} \right\}$$

is a dense subset of

$$(4.35) \quad \left\{ \begin{array}{l} (\phi, \psi) \in L^2(0, T; \mathbb{R}^{n_c}) \times W^{(q),2}(0, T) : \\ \phi_{i,t} = 0, \quad t \in \Delta_{c,i}^0, \quad 1 \leq i \leq n_c \\ \psi_{i,t} = 0, \quad t \in \Delta_{g,i}^0, \quad 1 \leq i \leq n_g \end{array} \right\}.$$

Indeed, for $\phi_i \in L^2(0, T)$, we consider the sequence

$$(4.36) \quad \phi_{i,t}^k := \begin{cases} 0 & \text{if } t \in \Delta_{c,i}^{1/k}, \\ \min\{k, |\phi_{i,t}|\} \frac{\phi_{i,t}}{|\phi_{i,t}|} & \text{otherwise.} \end{cases}$$

For $\psi_i \in W^{q_i,2}(0, T)$, we use the fact that there is no isolated point in $\Delta_{g,i}^0$ if $q_i \geq 2$, and approximation results in $W^{q_i,2}(0, T)$, e.g., [2, Appendix A.3]. Our claim follows.

(b) By Assumption 5 and the open mapping theorem, there exists $C > 0$ such that for all $(\phi, \psi) \in L^2(0, T; \mathbb{R}^{n_c}) \times W^{(q),2}(0, T)$, there exists $(v, z) \in \mathcal{V}_2 \times \mathcal{Z}_2$ such that

$$(4.37) \quad z = z[v, z_0], \quad \|v\|_2 + |z_0| \leq C (\|\phi\|_2 + \|\psi\|_{(q),2}),$$

$$(4.38) \quad \begin{aligned} Dc_i(t, \bar{u}_t, \bar{y}_t)(v_t, z_t) &= \phi_{i,t}, & t \in \Delta_{c,i}^{\delta'}, & \quad 1 \leq i \leq n_c, \\ Dg_i(t, \bar{y}_t)z_t &= \psi_{i,t}, & t \in \Delta_{g,i}^{\varepsilon'}, & \quad 1 \leq i \leq n_g. \end{aligned}$$

It follows that the subspace

$$(4.39) \quad \left\{ \begin{array}{l} (v, z) \in \mathcal{V}_2 \times \mathcal{Z}_2 : z = z[v, z_0] \\ \exists \delta > 0 : Dc_i(t, \bar{u}_t, \bar{y}_t)(v_t, z_t) = 0, \quad t \in \Delta_{c,i}^\delta, \quad 1 \leq i \leq n_c \\ \exists M > 0 : |Dc_i(t, \bar{u}_t, \bar{y}_t)(v_t, z_t)| \leq M, \quad t \in \Delta_{c,i}^{\delta'}, \quad 1 \leq i \leq n_c \\ \exists \varepsilon > 0 : Dg_i(t, \bar{y}_t)z_t = 0, \quad t \in \Delta_{g,i}^\varepsilon, \quad 1 \leq i \leq n_g \end{array} \right\}$$

is a dense subset of

$$(4.40) \quad \left\{ \begin{array}{l} (v, z) \in \mathcal{V}_2 \times \mathcal{Z}_2 : z = z[v, z_0] \\ Dc_i(t, \bar{u}_t, \bar{y}_t)(v_t, z_t) = 0, \quad t \in \Delta_{c,i}^0, \quad 1 \leq i \leq n_c \\ Dg_i(t, \bar{y}_t)z_t = 0, \quad t \in \Delta_{g,i}^0, \quad 1 \leq i \leq n_g \end{array} \right\}.$$

Observe now that $C_2^R(\bar{u}, \bar{v})$ and $C_2^S(\bar{u}, \bar{v})$ are defined by (4.33) and (4.17), respectively, as the same polyhedral cone in the previous two vector spaces; see also Remark 4.8.2. Then by [15, Lemma 1], the conclusion of Proposition 4.19 follows. \square

The definition of the set $C_2^{QR}(P_N^R)$ of quasi-radial critical directions in L^2 is given in Appendix A. Recall that $(\bar{\omega}, \bar{\vartheta}) := (0, 0) \in \mathcal{A}^N \times \mathbb{R}$.

LEMMA 4.20. *Let $(v, z) \in C_2^R(\bar{u}, \bar{y})$. Then $(v, \bar{\omega}, z_0, \bar{\vartheta}) \in C_2^{QR}(P_N^R)$.*

Proof. The direction $(v, \bar{\omega}, z_0, \bar{\vartheta})$ is radial [9, Definition 3.52] for the finite-dimensional constraints, which are polyhedral, as well as for the constraints on α . Let δ and $M > 0$ be given by definition of $C_2^R(\bar{u}, \bar{y})$. Then for any $\sigma > 0$

$$(4.41) \quad c_i(t, \bar{u}_t, \bar{y}_t) + \sigma Dc_i(t, \bar{u}_t, \bar{y}_t)(v_t, z_t) \leq \begin{cases} 0 & \text{for a.a. } t \in \Delta_{c,i}^\delta, \\ -\delta + \sigma M & \text{for a.a. } t \in \Delta_{c,i}^{\delta'} \setminus \Delta_{c,i}^\delta, \end{cases}$$

i.e., $(v, \bar{\omega}, z_0, \bar{\vartheta})$ is radial for the constraint (4.27). The same argument holds for constraint (4.28) since there exists $\delta_0 > 0$ such that $g_i(t, \bar{y}_t) \leq -\delta_0$ for all $t \in \Delta_{g,i}^{\epsilon'} \setminus \Delta_{g,i}^\epsilon$. Then $(v, \bar{\omega}, z_0, \bar{\vartheta})$ is radial, and a fortiori quasi radial. \square

We conclude with a remark about a flaw and some perspectives.

Remark 4.21.

1. In the proof of the density result [6, Lemma 6.4(ii)], there is no reason that v^n belongs to L^∞ , and not only to L^2 , since $(v^n - v)$ is obtained as a preimage of $(w^n - w, \omega^n - \omega)$. The lemma is actually true but its proof requires some effort; see [2, Lemma 4.5] for the case without mixed constraints. The difficulty is avoided here because we do not have to show the density of an L^∞ cone, thanks to our abstract second-order necessary conditions, Theorem A.8, that are derived directly in L^2 .
2. The partial relaxation technique used here could very easily be extended to optimal control problems of differential equations with memory, and provide second-order optimality conditions in Pontryagin form. The main difficulty would concern pure state constraints; see [2] for the case of integral equations.
3. Since second-order sufficient conditions in Pontryagin form have been obtained [4], a sensitivity analysis of such problems in term of Pontryagin multipliers might be possible, in the same spirit as the article [8], which deals with problems with final-state constraints.

Appendix A. Abstract optimization results. In this section, we recall necessary conditions satisfied by a weak minimum of a general optimal control problem. These conditions have been used in this paper to prove our necessary conditions in Pontryagin form, namely, Theorems 3.1 and 4.9, via the partial relaxation, i.e., Lemmas 3.9 and 4.16.

We actually state and prove first- and second-order necessary conditions for a more abstract optimization problem. It has to be noted that our second-order conditions, Theorem A.8, are obtained directly on a large set of directions in L^2 , thanks to metric regularity result, Lemma A.10, and a tricky truncation, Lemma A.11. To our knowledge, this is new.

A.1. Setting. Let K be a nonempty closed convex subset of a Banach space X and $\Delta_1, \dots, \Delta_M$ be measurable sets of $[0, T]$. For $s \in [1, \infty]$, let

$$(A.1) \quad \mathcal{U}_s := L^s(0, T; \mathbb{R}^{\bar{m}}), \quad \mathcal{Y}_s := W^{1,s}(0, T; \mathbb{R}^{\bar{n}}),$$

$$(A.2) \quad X_s := X \times \prod_{i=1}^M L^s(\Delta_i), \quad K_s := K \times \prod_{i=1}^M L^s(\Delta_i; \mathbb{R}_-).$$

We consider

$$(A.3) \quad \Gamma: \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}} \rightarrow \mathcal{Y}_\infty, \quad J: \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R},$$

$$(A.4) \quad G_1: \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}} \rightarrow X, \quad G_2^i: \mathcal{U}_\infty \times \mathcal{Y}_\infty \rightarrow L^\infty(\Delta_i),$$

the last mappings being defined for $i = 1, \dots, M$ by

$$(A.5) \quad G_2^i(u, y)_t := m_i(t, u_t, y_t)$$

for a.a. $t \in \Delta_i$, where $m_i: [0, T] \times \mathbb{R}^{\bar{m}} \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}$. Let

$$(A.6) \quad G: \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}} \rightarrow X_\infty, \quad G(u, y^0) := (G_1(u, y^0), G_2(u, \Gamma(u, y^0))).$$

The optimization problem we consider is the following:

$$(AP) \quad \min_{(u, y^0) \in \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}}} J(u, y^0); \quad G(u, y^0) \in K_\infty.$$

Remark A.1. Optimal control problems fit into this framework as follows: given a uniformly quasi- C^1 mapping $F: \mathbb{R} \times \mathbb{R}^{\bar{m}} \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}^{\bar{n}}$ and the state equation

$$(A.7) \quad \dot{y}_t = F(t, u_t, y_t) \quad \text{for a.a. } t \in (0, T),$$

we define $\Gamma(u, y^0)$ as the unique $y \in \mathcal{Y}_\infty$ such that (A.7) holds and $y_0 = y^0$ for any $(u, y^0) \in \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}}$ such that it exists (for instance, in a neighborhood of such a (\bar{u}, \bar{y}^0)); given a cost function $\tilde{J}: \mathcal{Y}_\infty \rightarrow \mathbb{R}$, we define $J := \tilde{J} \circ \Gamma$; given state constraints of any kind (pure, initial-final, ...) $\tilde{G}_1: \mathcal{Y}_\infty \rightarrow X$, with the appropriate space X and convex subset K , we define $G_1 := \tilde{G}_1 \circ \Gamma$; finally, we define G_2 in order to take into account the mixed control-state and control constraints. By definition, a weak minimum of such an optimal control problem is a locally optimal solution of the corresponding optimization problem (AP).

A.2. Assumptions. Let (\bar{u}, \bar{y}^0) be feasible for (AP) and let $\bar{y} := \Gamma(\bar{u}, \bar{y}^0)$. For various Banach spaces Y and mappings $\mathcal{F}: \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}} \rightarrow Y$, we will require one of the following properties.

PROPERTY A.2. *The mapping \mathcal{F} is C^1 in a neighborhood of (\bar{u}, \bar{y}^0) , with continuous extensions $D\mathcal{F}(u, y^0): \mathcal{U}_1 \times \mathbb{R}^{\bar{n}} \rightarrow Y$.*

PROPERTY A.3. *Property A.2 holds, and \mathcal{F} is twice Fréchet differentiable at (\bar{u}, \bar{y}^0) , with a continuous extension $D^2\mathcal{F}(\bar{u}, \bar{y}^0): (\mathcal{U}_2 \times \mathbb{R}^{\bar{n}})^2 \rightarrow Y$ and the following expansion in Y : for all $(v, z^0) \in \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}}$,*

$$(A.8) \quad \mathcal{F}(\bar{u} + v, \bar{y}^0 + z^0) = \mathcal{F}(\bar{u}, \bar{y}^0) + D\mathcal{F}(\bar{u}, \bar{y}^0)(v, z^0) + \frac{1}{2}D^2\mathcal{F}(\bar{u}, \bar{y}^0)(v, z^0)^2 + \mathcal{R}(\|v\|_\infty + |z^0|)(\|v\|_2^2 + |z^0|^2),$$

where $\mathcal{R}(\alpha) \rightarrow 0$ in Y as $\alpha \rightarrow 0$.

Assumption 6. The mappings Γ , J , and G_1 satisfy Property A.2, and the functions m_i are uniformly quasi- C^1 .

Assumption 6'. The mappings Γ , J , and G_1 satisfy Property A.3, and the functions m_i are uniformly quasi- C^2 .

Assumption 7. Robinson's constraint qualification holds:

$$(A.9) \quad 0 \in \text{int}_{X_\infty} \{G(\bar{u}, \bar{y}^0) + DG(\bar{u}, \bar{y}^0)(\mathcal{U}_\infty \times \mathbb{R}^{\bar{n}}) - K_\infty\}.$$

Assumption 8. The inward condition holds for G_2 : there exists $\gamma > 0$ and $\hat{v} \in \mathcal{U}_\infty$ such that

$$(A.10) \quad G_2^i(\bar{u}, \bar{y}) + D_u G_2^i(\bar{u}, \bar{y})\hat{v} \leq -\gamma \quad \text{on } \Delta_i \quad i = 1, \dots, M.$$

Remark A.4. Let us consider the case of an optimal control problem, with Γ, J , and G_1 defined as in Remark A.1. If F, m_i are uniformly quasi- C^1 and \tilde{J}, \tilde{G}_1 are C^1 , then Assumption 6 holds. If F, m_i are uniformly quasi- C^2 and \tilde{J}, \tilde{G}_1 are C^2 , then Assumption 6' holds. See, for example, [5, Lemmas 19–20] or [31, Theorems 3.3–3.5].

LEMMA A.5. *Let Assumptions 6–8 hold. Then the following qualification condition is satisfied:*

$$(A.11) \quad 0 \in \text{int}_{X_1} \{G(\bar{u}, \bar{y}^0) + DG(\bar{u}, \bar{y}^0)(\mathcal{U}_1 \times \mathbb{R}^{\bar{n}}) - T_{K_1}(G(\bar{u}, \bar{y}^0))\}.$$

Proof. Let $(x_1, x_2^1, \dots, x_2^N) \in X \times \prod_{i=1}^N L^1(\Delta_i)$, let us define for a.a. t ,

$$(A.12) \quad \varepsilon_t = \max \left\{ \max_{i=1, \dots, N} \left\{ \frac{-x_2^i, t}{\gamma} : t \in \Delta_i, m_i(t, \bar{u}_t, \bar{y}_t) = 0, x_2^i \leq 0 \right\}; 0 \right\}.$$

We set $v_t = \varepsilon_t \hat{v}_t$ for a.a. t and define $\tilde{x}_2 = D_u G_2(\bar{u}, \bar{y})v - x_2$. It is easy to check that for all i , $\tilde{x}_2^i \in T_{L^1(\Delta_i; \mathbb{R}_-)}(G_2^i(\bar{u}, \bar{y}))$ and, therefore, $(0, \tilde{x}_2) \in T_{K_1}(G(\bar{u}, \bar{y}^0))$. Set

$$(A.13) \quad (x'_1, x'_2) = (x_1, 0) - (D_u G_1(\bar{u}, \bar{y}^0)v, D_y G_2(\bar{u}, \bar{y})D_u \Gamma(\bar{u}, \bar{y}^0))$$

By Property A.2, $\|(x'_1, x'_2)\|_{X_\infty} = O(\|(x_1, x_2)\|_{X_1})$. Therefore, if $\|(x_1, x_2)\|_{X_1}$ is sufficiently small, there exist $(w, \xi^0) \in \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}}$ and $(\hat{x}_1, \hat{x}_2) \in K_\infty$ such that

$$(A.14) \quad (x'_1, x'_2) = G(\bar{u}, \bar{y}^0) + DG(\bar{u}, \bar{y}^0)(w, \xi^0) - (\hat{x}_1, \hat{x}_2).$$

Finally, $(x_1, x_2) = G(\bar{u}, \bar{y}^0) + DG(\bar{u}, \bar{y}^0)(w + v, \xi^0) - (\hat{x}_1, \hat{x}_2 + \tilde{x}_2)$, which proves the lemma, since by convexity of K , $(\hat{x}_1, \hat{x}_2) \in T_{K_\infty}(G(\bar{u}, \bar{y}^0)) \subset T_{K_1}(G(\bar{u}, \bar{y}^0))$. \square

A.3. Necessary conditions. We consider the *Lagrangian* $L[\lambda]: \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}$, defined for $\lambda \in X_\infty^*$ by

$$(A.15) \quad L[\lambda](u, y^0) := J(u, y^0) + \langle \lambda, G(u, y^0) \rangle.$$

We define the set of *Lagrange multipliers* as

$$(A.16) \quad \Lambda(AP) := \{ \lambda \in X_1^* : \lambda \in N_{K_1}(G(\bar{u}, \bar{y}^0)), DL[\lambda](\bar{u}, \bar{y}^0) = 0 \text{ on } \mathcal{U}_1 \times \mathbb{R}^{\bar{n}} \},$$

and the set of *quasi-radial critical directions* in L^2 as

$$(A.17) \quad C_2^{QR}(AP) := \left\{ (v, z^0) \in \mathcal{U}_2 \times \mathbb{R}^{\bar{n}} : DJ(\bar{u}, \bar{y}^0)(v, z^0) \leq 0 \text{ and } \forall \sigma > 0, \right. \\ \left. \text{dist}_{X_1}(G(\bar{u}, \bar{y}^0) + \sigma DG(\bar{u}, \bar{y}^0)(v, z^0), K_1) = o(\sigma^2) \right\}.$$

We denote by $\text{cl}(C_2^{QR}(AP))$ its closure in $\mathcal{U}_2 \times \mathbb{R}^{\bar{n}}$.

Remark A.6. If $(v, z^0) \in C_2^{QR}(AP)$, then $DG(\bar{u}, \bar{y}^0)(v, z^0) \in T_{K_1}(G(\bar{u}, \bar{y}^0))$. If in addition $\Lambda(AP) \neq \emptyset$, then $DJ(\bar{u}, \bar{y}^0)(v, z^0) = 0$.

We now state our first- and second-order necessary conditions, in two theorems that will be proved in the next section.

THEOREM A.7. *Let (\bar{u}, \bar{y}^0) be a locally optimal solution of (AP), and let Assumptions 6–8 hold. Then $\Lambda(AP)$ is nonempty, convex, and weakly * compact in X_1^* .*

THEOREM A.8. *Let (\bar{u}, \bar{y}^0) be a locally optimal solution of (AP), and let Assumptions 6'–8 hold. Then for any $(v, z^0) \in \text{cl}(C_2^{QR}(AP))$, there exists $\lambda \in \Lambda(AP)$ such that*

$$(A.18) \quad D^2 L[\lambda](\bar{u}, \bar{y}^0)(v, z^0)^2 \geq 0.$$

A.4. Proofs.

Proof of Theorem A.7. Robinson’s constraint qualification (A.9) and [32, Theorem 4.1] or [9, Theorem 3.9] give the result in X_∞^* . We derive it in X_1^* with the inward condition (Assumption 8); see, e.g., [7, Theorem 3.1]. \square

Proof of Theorem A.8. (a) Assume first that $(v, z^0) \in C_2^{QR}(AP)$. We consider the following conic linear problem [9, section 2.5.6]:

$$(Q_{(v,z^0)}) \quad \begin{cases} \min_{(w,\xi^0) \in \mathcal{U}_1 \times \mathbb{R}^n} DJ(\bar{u}, \bar{y}^0)(w, \xi^0) + D^2J(\bar{u}, \bar{y}^0)(v, z^0)^2 ; \\ DG(\bar{u}, \bar{y}^0)(w, \xi^0) + D^2G(\bar{u}, \bar{y}^0)(v, z^0)^2 \in T_{K_1}(G(\bar{u}, \bar{y}^0)). \end{cases}$$

By Lemma A.5 the constraints of $(Q_{(v,z^0)})$ are regular in the sense of [9, Theorem 2.187]. Then by the same theorem, there is no duality gap between $(Q_{(v,z^0)})$ and its dual, which is the following optimization problem:

$$(A.19) \quad \max_{\lambda \in \Lambda(AP)} D^2L[\lambda](\bar{u}, \bar{y}^0)(v, z^0)^2.$$

Observe indeed that the Lagrangian of $(Q_{(v,z^0)})$ is

$$(A.20) \quad \mathcal{L}[\lambda](w, \xi^0) = DL[\lambda](\bar{u}, \bar{y}^0)(w, \xi^0) + D^2L[\lambda](\bar{u}, \bar{y}^0)(v, z^0)^2, \lambda \in X_1^*.$$

The conclusion of the theorem follows when $(v, z^0) \in C_2^{QR}(AP)$ by the following key lemma, that will be proved below.

LEMMA A.9. *The value of $(Q_{(v,z^0)})$ is nonnegative.*

(b) Assume now that $(v, z^0) \in \text{cl}(C_2^{QR}(AP))$. Let $(v^k, z^{0,k}) \in C_2^{QR}(AP)$ converge to (v, z^0) in $\mathcal{U}_2 \times \mathbb{R}^n$. By step (a), there exists $\lambda^k \in \Lambda(AP)$ such that

$$(A.21) \quad \begin{aligned} D^2J(\bar{u}, \bar{y}^0)(v^k, z^{0,k})^2 + \langle \lambda^k, D^2G(\bar{u}, \bar{y}^0)(v^k, z^{0,k})^2 \rangle \\ = D^2L[\lambda^k](\bar{u}, \bar{y}^0)(v^k, z^{0,k})^2 \geq 0. \end{aligned}$$

By Theorem A.7, there exists $\lambda \in \Lambda(AP)$ such that, up to a subsequence, $\lambda^k \rightharpoonup \lambda$ for the weak $*$ topology in X_1^* . By Assumption 6', $D^2J(\bar{u}, \bar{y}^0): \mathcal{U}_2 \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $D^2G(\bar{u}, \bar{y}^0): \mathcal{U}_2 \times \mathbb{R}^n \rightarrow X_1$ are continuous. The conclusion follows. \square

Proof of Lemma A.9. First we prove a metric regularity result, which relies on Assumption 8. For any $(u, y) \in \mathcal{U}_\infty \times \mathcal{Y}_\infty$, we define $G_2^+(u, y) \in L^\infty(0, T)$ by

$$(A.22) \quad G_2^+(u, y)_t := \max_{1 \leq i \leq M} (G_2^i(u, y)_t)_+$$

for a.a. $t \in (0, T)$, where

$$(A.23) \quad (G_2^i(u, y)_t)_+ := \begin{cases} \max\{0, G_2^i(u, y)_t\} & \text{if } t \in \Delta_i, \\ 0 & \text{if } t \notin \Delta_i. \end{cases}$$

LEMMA A.10. *There exists $c > 0$ such that, for any $(u, y) \in \mathcal{U}_\infty \times \mathcal{Y}_\infty$ with $y = \Gamma(u, y^0)$ in a neighborhood of (\bar{u}, \bar{y}) , there exists $(\hat{u}, \hat{y}) \in \mathcal{U}_\infty \times \mathcal{Y}_\infty$ with $\hat{y} = \Gamma(\hat{u}, y^0)$ such that*

$$(A.24) \quad \|\hat{u} - u\|_\infty \leq c \|G_2^+(u, y)\|_\infty,$$

$$(A.25) \quad \|\hat{u} - u\|_1 \leq c \|G_2^+(u, y)\|_1,$$

$$(A.26) \quad \|G_2^+(\hat{u}, \hat{y})\|_\infty \leq c \|G_2^+(u, y)\|_1.$$

Proof. Let $\beta \in (0, 1)$ to be fixed later. Since (\bar{u}, \bar{y}^0) is feasible, $G_2^+(\bar{u}, \bar{y}) = 0$, and there exists $\alpha \in (0, \beta)$ such that

$$(A.27) \quad \|u - \bar{u}\|_\infty + \|y - \bar{y}\|_\infty \leq \alpha \Rightarrow \|G_2^+(u, y)\|_\infty \leq \beta.$$

Let (u, y) be such that $\|u - \bar{u}\|_\infty + \|y - \bar{y}\|_\infty \leq \alpha$. We define $\varepsilon \in L^\infty(0, T)$ by

$$(A.28) \quad \varepsilon_t := \frac{1}{\beta} G_2^+(u, y)_t,$$

so that $\varepsilon_t \in [0, 1]$ for a.a. $t \in (0, T)$, and

$$(A.29) \quad \hat{u} := u + \varepsilon \hat{v},$$

where \hat{v} is given by Assumption 8. Once β is fixed, it is clear that (A.24) and (A.25) hold. Let $\hat{y} = \Gamma(\hat{u}, y^0)$. For all i ,

$$(A.30) \quad \begin{aligned} G_2^i(\hat{u}, \hat{y}) &= G_2^i(u, y) + DG_2^i(\bar{u}, \bar{y})(\hat{u} - u, \hat{y} - y) \\ &+ \int_0^1 \left(DG_2^i(u + \theta(\hat{u} - u), y + \theta(\hat{y} - y)) \right. \\ &\quad \left. - DG_2^i(\bar{u}, \bar{y}) \right) (\hat{u} - u, \hat{y} - y) d\theta \end{aligned}$$

a.e. on Δ_i . Since Γ satisfies Property A.2, $\|\hat{y} - y\|_\infty = O(\|\hat{u} - u\|_1)$, and then

$$(A.31) \quad |\hat{u}_t - u_t| = O(\varepsilon_t), \quad |u_t - \bar{u}_t| = O(\alpha) = O(\beta),$$

$$(A.32) \quad |\hat{y}_t - y_t| = O(\|\varepsilon\|_1), \quad |y_t - \bar{y}_t| = O(\alpha) = O(\beta).$$

Since m_i is uniformly quasi- C^2 , G_2^i and DG_2^i are Lipschitz in a neighborhood of (\bar{u}, \bar{y}) . Then

$$(A.33) \quad G_2^i(\hat{u}, \hat{y}) = G_2^i(u, y) + \varepsilon D_u G_2^i(\bar{u}, \bar{y}) \hat{v} + O(\|\varepsilon\|_1 + \varepsilon(\varepsilon + \|\varepsilon\|_1 + \beta))$$

$$(A.34) \quad \begin{aligned} &= (1 - \varepsilon) G_2^i(u, y) + \varepsilon (G_2^i(u, y) - G_2^i(\bar{u}, \bar{y})) \\ &\quad + \varepsilon (G_2^i(\bar{u}, \bar{y}) + D_u G_2^i(\bar{u}, \bar{y}) \hat{v}) + O(\|\varepsilon\|_1 + \varepsilon(\varepsilon + \|\varepsilon\|_1 + \beta)). \end{aligned}$$

The previous equalities, as well as the following, are given for a.a. t in Δ_i , with remainders O which are uniform in time. Observe now that

$$(A.35) \quad (1 - \varepsilon) G_2^i(u, y) \leq G_2^+(u, y^0) = \varepsilon \beta,$$

$$(A.36) \quad \varepsilon (G_2^i(u, y) - G_2^i(\bar{u}, \bar{y})) = O(\alpha \varepsilon) = O(\varepsilon \beta),$$

$$(A.37) \quad \varepsilon (G_2^i(\bar{u}, \bar{y}) + D_u G_2^i(\bar{u}, \bar{y}) \hat{v}) \leq -\varepsilon \gamma.$$

Then there exists $C > 0$, independent of u and u' , such that

$$(A.38) \quad G_2^i(\hat{u}, \hat{y}) \leq C \|\varepsilon\|_1 + \varepsilon [C(\varepsilon + \|\varepsilon\|_1 + \beta) - \gamma]$$

on Δ_i , $i = 1, \dots, M$. We fix $\beta \in (0, 1)$ such that $C\beta \leq \gamma/2$ and $\alpha \in (0, \beta)$ such that $C(\varepsilon + \|\varepsilon\|_1) \leq \gamma/2$. The result follows. \square

To prove Lemma A.9, we also need the following.

LEMMA A.11. *Let $v \in \mathcal{U}_2$ and $w \in \mathcal{U}_1$. Let $v^k := \mathbf{1}_{\{|v| \leq k\}} v$, $w^k := \mathbf{1}_{\{|w| \leq k\}} w$, and $\sigma_k := \frac{\|v^k - v\|_2}{k}$. Then $v^k, w^k \in \mathcal{U}_\infty$, $\sigma_k \rightarrow 0$, and*

$$(A.39) \quad \|\sigma_k v^k\|_\infty = o(1), \quad \|\sigma_k^2 w^k\|_\infty = o(1),$$

$$(A.40) \quad \|v^k - v\|_2 = o(1), \quad \|w^k - w\|_1 = o(1),$$

$$(A.41) \quad \|v^k - v\|_1 = o(\sigma_k).$$

Proof. We first get (A.40) by Lebesgue’s dominated convergence theorem. Then $\sigma_k = o(\frac{1}{k})$, and (A.39) follows. Observe that $|v^k - v|^2 \geq k|v^k - v|$, which implies $\|v^k - v\|_1 = O(\frac{1}{k}\|v^k - v\|_2^2)$. (A.41) follows by definition of σ_k and by (A.40). \square

Let us now go back to the proof of Lemma A.9: let (w, ξ^0) be feasible for problem $(Q_{(v, z^0)})$. We apply Lemma A.11 to $v \in \mathcal{U}_2$, $w \in \mathcal{U}_1$, and we consider

$$(A.42) \quad u^k := \bar{u} + \sigma_k v^k + \frac{1}{2}\sigma_k^2 w^k \in \mathcal{U}_\infty,$$

$$(A.43) \quad y^{0,k} := \bar{y}^0 + \sigma_k z^0 + \frac{1}{2}\sigma_k^2 \xi^0 \in \mathbb{R}^{\bar{n}},$$

$$(A.44) \quad y^k := \Gamma(u^k, y^{0,k}) \in \mathcal{Y}_\infty.$$

We have in particular

$$(A.45) \quad \|u^k - \bar{u}\|_\infty = o(1), \quad \|u^k - \bar{u}\|_2 = O(\sigma_k).$$

By analogy with linearized trajectories, we denote

$$(A.46) \quad z[\tilde{v}, \tilde{z}^0] := D\Gamma(\bar{u}, \bar{y}^0)(\tilde{v}, \tilde{z}^0), \quad z^2[\tilde{v}, \tilde{z}^0] := D^2\Gamma(\bar{u}, \bar{y}^0)(\tilde{v}, \tilde{z}^0)^2$$

for any $(\tilde{v}, \tilde{z}^0) \in \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}}$. Since Γ satisfies Property A.3, we have in \mathcal{Y}_∞

$$(A.47) \quad y^k = \bar{y} + \sigma_k z[v^k, z^0] + \frac{1}{2}\sigma_k^2 (z[w^k, \xi^0] + z^2[v^k, z^0]) + o(\sigma_k^2),$$

and in particular, $\|y^k - \bar{y}\|_\infty = O(\sigma_k)$. Then $(u^k, y^k) \rightarrow (\bar{u}, \bar{y})$ in $\mathcal{U}_\infty \times \mathcal{Y}_\infty$ and

$$(A.48) \quad \|G_2^+(u^k, y^k)\|_\infty = o(1).$$

More precisely, since m_i is uniformly quasi- C^2 , we have

$$(A.49) \quad G_2^i(u^k, y^k) = G_2^i(\bar{u}, \bar{y}) + DG_2^i(\bar{u}, \bar{y})(u^k - \bar{u}, y^k - \bar{y}) \\ + \frac{1}{2}D^2G_2^i(\bar{u}, \bar{y})(u^k - \bar{u}, y^k - \bar{y})^2 + o(|u^k - \bar{u}|^2 + |y^k - \bar{y}|^2)$$

a.e. on Δ_i , where $o(\cdot)$ is uniform w.r.t. t . We write

$$(A.50) \quad G_2^i(u^k, y^k)_t = \frac{1}{2}T_t^{i,k} + \frac{1}{2}Q_t^{i,k} + R_t^{i,k},$$

where, omitting the time argument t ,

$$(A.51) \quad T^{i,k} := G_2^i(\bar{u}, \bar{y}) + 2\sigma_k DG_2^i(\bar{u}, \bar{y})(v^k, z[v^k, z^0]),$$

$$Q^{i,k} := G_2^i(\bar{u}, \bar{y}) + \sigma_k^2 (DG_2^i(\bar{u}, \bar{y})(w^k, z[w^k, \xi^0])$$

$$(A.52) \quad + D^2G_2^i(\bar{u}, \bar{y})(v^k, z[v^k, z^0])^2 + D_y G_2^i(\bar{u}, \bar{y})z^2[v^k, z^0]),$$

$$R^{i,k} = \frac{1}{2}\sigma_k^3 D^2G_2^i(\bar{u}, \bar{y}) [(v^k, z[v^k, z^0]), (w^k, z[w^k, \xi^0] + z^2[v^k, z^0] + o(1))]$$

$$(A.53) \quad + \frac{1}{4}\sigma_k^4 D^2G_2^i(\bar{u}, \bar{y})(w^k, z[w^k, \xi^0] + z^2[v^k, z^0] + o(1))^2$$

$$+ o(|u^k - \bar{u}|^2 + |y^k - \bar{y}|^2).$$

We claim that $\|R^{i,k}\|_1 = o(\sigma_k^2)$. Indeed, $z[v^k, z^0]$, $z[w^k, \xi^0]$, and $z^2[v^k, z^0]$ are bounded in \mathcal{Y}_∞ ; the crucial terms are then the following:

$$(A.54) \quad \|\sigma_k^3 D_{uu}^2 G_2^i(\bar{u}, \bar{y})(v^k, w^k)\|_1 = O(\|\sigma_k v^k\|_\infty \cdot \|\sigma_k^2 w^k\|_1) = o(\sigma_k^2),$$

$$(A.55) \quad \|\sigma_k^4 D_{uu}^2 G_2^i(\bar{u}, \bar{y})(w^k, w^k)\|_1 = O(\|\sigma_k^2 w^k\|_\infty \cdot \|\sigma_k^2 w^k\|_1) = o(\sigma_k^2),$$

$$(A.56) \quad \|o(|u^k - \bar{u}|^2 + |y^k - \bar{y}|^2)\|_1 = o(\|u^k - \bar{u}\|_2^2 + \|y^k - \bar{y}\|_2^2) = o(\sigma_k^2)$$

by (A.39), (A.40) and (A.45), (A.47). Recall that $(v, z^0) \in C_2^{QR}(AP)$. Then by (A.41) and Property A.2, satisfied by Γ , we have

$$(A.57) \quad \text{dist}_{L^1}(T^{i,k}, L^1(\Delta_i; \mathbb{R}_-)) = o(\sigma_k^2).$$

Similarly, since (w, ξ^0) is feasible for $(Q_{(v, z^0)})$ and Γ satisfies Property A.3,

$$(A.58) \quad \text{dist}_{L^1}(Q^{i,k}, L^1(\Delta_i; \mathbb{R}_-)) = o(\sigma_k^2).$$

Then, in addition to (A.48), we have proved that

$$(A.59) \quad \|G_2^+(u^k, y^k)\|_1 = o(\sigma_k^2).$$

We apply now Lemma A.10 to the sequence (u^k, y^k) ; we get a sequence $(\hat{u}^k, \hat{y}^k) \in \mathcal{U}_\infty \times \mathcal{Y}_\infty$ with $\hat{y}^k = \Gamma(\hat{u}^k, y^{0,k})$ and such that

$$(A.60) \quad \|\hat{u}^k - u^k\|_\infty = o(1),$$

$$(A.61) \quad \|\hat{u}^k - u^k\|_1 = o(\sigma_k^2),$$

$$(A.62) \quad \|G_2^+(\hat{u}^k, \hat{y}^k)\|_\infty = o(\sigma_k^2).$$

Since G_1 satisfies Property A.3, $(v, z^0) \in C_2^{QR}(AP)$ and (w, ξ^0) is feasible for $(Q_{(v, z^0)})$, we get

$$(A.63) \quad \text{dist}_X(G_1(\hat{u}^k, y^{0,k}), K) = o(\sigma_k^2),$$

and then, together with (A.62),

$$(A.64) \quad \text{dist}_{X_\infty}(G(\hat{u}^k, y^{0,k}), K_\infty) = o(\sigma_k^2).$$

By Robinson's constraint qualification (A.9), G is metric regular at (\bar{u}, \bar{y}^0) w.r.t. K_∞ , [9, Theorem 2.87]. Then there exists $(\tilde{u}^k, \tilde{y}^{0,k}) \in \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}}$ such that

$$(A.65) \quad \begin{cases} \|\tilde{u}^k - \hat{u}^k\|_\infty + |\tilde{y}^{0,k} - y^{0,k}| = o(\sigma_k^2), \\ G(\tilde{u}^k, \tilde{y}^{0,k}) \in K_\infty. \end{cases}$$

Since (\bar{u}, \bar{y}^0) is a locally optimal solution, $J(\tilde{u}^k, \tilde{y}^{0,k}) \geq J(\bar{u}, \bar{y}^0)$ for k big enough. By Property A.3, satisfied by J , we have

$$(A.66) \quad \sigma_k DJ(\bar{u}, \bar{y}^0)(v, z^0) + \frac{1}{2} \sigma_k^2 (DJ(\bar{u}, \bar{y}^0)(w, \xi^0) + D^2 J(\bar{u}, \bar{y}^0)(v, z^0)^2) + o(\sigma_k^2) \geq 0.$$

The conclusion of Lemma A.9 follows by Theorem A.7 and Remark A.6. \square

Appendix B. A qualification condition.

B.1. Statement. We give here a qualification condition equivalent to the non-singularity of generalized Pontryagin multipliers. This qualification condition is expressed with the Pontryagin linearization [25, Proposition 8.1]. In this section, $(\bar{u}, \bar{y}) \in F(P)$ is given. We will always assume that Assumption 2 holds.

DEFINITION B.1. We say that $\lambda = (\beta, \Psi, \nu, \mu) \in \Lambda_L(\bar{u}, \bar{y})$ is singular iff $\beta = 0$ and that λ is normal iff $\beta = 1$.

Given $u \in \mathcal{U}$, we define the Pontryagin linearization $\xi[u] \in \mathcal{Y}$ as the unique solution of

$$(B.1) \quad \begin{cases} \dot{\xi}_t[u] = D_y f(t, \bar{u}_t, \bar{y}_t) \xi_t[u] + f(t, u_t, \bar{y}_t) - f(t, \bar{u}_t, \bar{y}_t), \\ \xi_0[u] = 0. \end{cases}$$

Note that $\xi[\bar{u}] = 0$. Recall that U is the set-valued mapping defined by (2.20). We define

$$(B.2) \quad \mathcal{U}_c := \{u \in \mathcal{U} : u_t \in U(t) \text{ for a.a. } t\}.$$

DEFINITION B.2. We say that the problem is qualified in the Pontryagin sense (in short P-qualified) at (\bar{u}, \bar{y}) iff

(i) the following surjectivity condition holds:

$$(B.3) \quad 0 \in \text{int} \{ D\Phi^E(\bar{y}_0, \bar{y}_T)(z_0, \xi_T[u] + z_T[v, z_0]) : u \in \mathcal{U}_c, v \in \mathcal{U}, z_0 \in \mathbb{R}^n \};$$

(ii) there exist $\varepsilon > 0$, $\hat{u} \in \mathcal{U}_c$, $\hat{v} \in \mathcal{U}$, and $\hat{z}_0 \in \mathbb{R}^n$ such that

$$(B.4) \quad D\Phi^E(\bar{y}_0, \bar{y}_T)(\hat{z}_0, \xi_T[\hat{u}] + z_T[\hat{v}, \hat{z}_0]) = 0,$$

and for a.a. t ,

$$(B.5) \quad \begin{cases} \Phi^I(\bar{y}_0, \bar{y}_T) + D\Phi^I(\bar{y}_0, \bar{y}_T)(\hat{z}_0, \xi_T[\hat{u}] + z_T[\hat{v}, \hat{z}_0]) \leq -\varepsilon, \\ g(t, \bar{y}_t) + Dg(t, \bar{y}_t)(\xi_t[\hat{u}] + z_t[\hat{v}, \hat{z}_0]) \leq -\varepsilon, \\ c(t, \bar{u}_t, \bar{y}_t) + D_y c(t, \bar{u}_t, \bar{y}_t) \xi_t[\hat{u}] + Dc(t, \bar{u}_t, \bar{y}_t)(\hat{v}_t, z_t[\hat{v}_t, \hat{z}_0]) \leq -\varepsilon. \end{cases}$$

Note that if we impose $u = \bar{u}$ in the definition of the P-qualification, we obtain the usual qualification conditions, which are equivalent to the normality of Lagrange multipliers. The P-qualification is then weaker, and as proved in the next theorem, it is necessary and sufficient to ensure the nonsingularity of Pontryagin multipliers.

THEOREM B.3. Let Assumption 2 hold. Then, the set of singular Pontryagin multipliers is empty iff the problem is P-qualified.

We prove this result in the following two paragraphs.

PROPOSITION B.4. Let Assumption 2 hold. If the set of singular Pontryagin multipliers is empty, then the set of normal Pontryagin multipliers is bounded in E .

Proof. Remember that the norm of E is defined by (2.11). We prove the result by contraposition and consider a sequence $(\lambda^k)_k$ of normal Pontryagin multipliers which is such that $\|\lambda^k\|_E \rightarrow +\infty$. Then, by Lemma 3.5, the sequence $\lambda^k / \|\lambda^k\|_E$ possesses a weak limit point in $\Lambda_P(\bar{u}, \bar{y})$, say $\lambda = (\beta, \Psi, \nu, \mu)$, which is such that

$$(B.6) \quad \beta = \lim_k \frac{1}{\|\lambda^k\|_E} = 0.$$

Therefore, λ is singular. The proposition is proved. \square

B.2. Sufficiency of the qualification condition. In this paragraph, we prove by contradiction that the P-qualification implies the nonsingularity of Pontryagin multipliers. Let us assume that the problem is P-qualified and that there exists $\lambda = (\beta, \Psi, \nu, \mu) \in \Lambda_P(\bar{u}, \bar{y})$ with $\beta = 0$ and $\Psi = (\Psi^E, \Psi^I)$. Let $\hat{u}, \hat{v}, \hat{z}_0$ be such that (B.4)–(B.5) hold. With an integration by parts and using the stationarity of the augmented Hamiltonian, we get that for all $u \in \mathcal{U}_c, v \in \mathcal{U}$, and $z_0 \in \mathbb{R}^n$,

$$\begin{aligned}
 & \int_0^T \nu_t (Dc(t, \bar{u}_t, \bar{y}_t)(v_t, z_t[v, z_0]) + D_y c(t, \bar{u}_t, \bar{y}_t) \xi_t[u]) dt \\
 & + \int_0^T Dg(t, \bar{y}_t)(\xi_t[u] + z_t[v, z_0]) d\mu_t \\
 & + D\Phi[0, (\Psi^E, \Psi^I)](\bar{y}_0, \bar{y}_T)(z_0, \xi_T[u] + z_T[v, z_0]) \\
 \text{(B.7)} \quad & = \int_0^T H[p_t^\lambda](t, u_t, \bar{y}_t) - H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) dt \geq 0.
 \end{aligned}$$

By (B.4)–(B.5) and the nonnegativity of Ψ^I, ν , and μ , we obtain that for $u = \hat{u}, v = \hat{v}, z_0 = \hat{z}_0$, the right-hand side of (B.7) is nonpositive and thus equal to 0. Therefore, Ψ^I, ν , and μ are null and for all $u \in \mathcal{U}_c, v \in \mathcal{U}$, and $z_0 \in \mathbb{R}^n$,

$$\text{(B.8)} \quad \Psi^E D\Phi^E(\bar{y}_0, \bar{y}_T)(z_0, \xi_T[u] + z_T[v, z_0]) \geq 0.$$

By (B.3), we can choose u, v , and z_0 so that for $\delta > 0$ sufficiently small,

$$\text{(B.9)} \quad D\Phi^E(\bar{y}_0, \bar{y}_T)(z_0, \xi_T[u] + z_T[v, z_0]) = -\delta(\Psi^E)^*.$$

Combined with (B.8), we obtain that $-\delta|\Psi^E|^2 \geq 0$. Then, $\Psi^E = 0$ and finally $\lambda = 0$, in contradiction to $\lambda \in \Lambda_P(\bar{u}, \bar{y})$.

B.3. Necessity of the qualification condition. We now prove that the P-qualification is necessary to ensure the nonsingularity of Pontryagin multipliers. In some sense, the approach consists in describing this qualification condition as the limit of the qualification conditions associated with a sequence of partially relaxed problems.

Let us fix a Castaing representation $(u^k)_k$ of U . For all $N \in \mathbb{N}$, we consider a partially relaxed problem (\tilde{P}_N) defined by

$$\text{(\tilde{P}_N)} \quad \min_{u \in \mathcal{U}, \alpha \in \mathcal{A}^N, y \in \mathcal{Y}} \phi(y_0, y_T) \text{ subject to (2.3)–(2.5), } y = y[u, \alpha, y_0], \alpha \geq 0,$$

where $y[u, \alpha, y^0]$ is the solution to the partially relaxed state equation (3.12). This problem is the same as problem (P_N) , except that there is no variable θ .

For given $v \in \mathcal{U}, z_0 \in \mathbb{R}^n$, and $\alpha \in \mathcal{A}^N$, we denote by $z[v, z_0]$ the linearized state variable in the direction (v, z_0) , which is the solution to (4.2) and we denote by $\xi[\alpha]$ the linearized state variable in the direction α , which is the solution to

$$\text{(B.10)} \quad \begin{cases} \dot{\xi}_t[\alpha] = D_y f(t, \bar{u}_t, \bar{y}_t) \xi_t[\alpha] + \sum_{i=1}^N \alpha_t^i (f(t, u_t^i, \bar{y}_t) - f(t, \bar{u}_t, \bar{y}_t)), \\ \xi_0[\alpha] = 0. \end{cases}$$

The distinction between the Pontryagin linearization $\xi[u]$ and $\xi[\alpha]$ will be clear in the sequel, and we will motivate this choice of notation in Lemma B.7.

Problem (\tilde{P}_N) is qualified (in the usual sense) iff

(i) the following surjectivity condition holds:

$$(B.11) \quad 0 \in \text{int}\{D\Phi^E(\bar{y}_0, \bar{y}_T)(z_0, \xi_T[\alpha] + z_T[v, z_0]) : \alpha \in \mathcal{A}^N, v \in \mathcal{U}, z_0 \in \mathbb{R}^n\};$$

(ii) there exist $\varepsilon > 0, \hat{\alpha} \in \mathcal{A}^N, \hat{v} \in \mathcal{U}, \hat{z}_0 \in \mathbb{R}^n$ such that

$$(B.12) \quad D\Phi^E(\bar{y}_0, \bar{y}_T)(\hat{z}_0, \xi_T[\hat{\alpha}] + z_T[\hat{v}, \hat{z}_0]) = 0$$

and for a.a. t ,

$$(B.13) \quad \begin{cases} \Phi^I(\bar{y}_0, \bar{y}_T) + D\Phi^I(\bar{y}_0, \bar{y}_T)(\hat{z}_0, \xi_T[\hat{\alpha}] + z_T[\hat{v}, \hat{z}_0]) \leq -\varepsilon, \\ g(t, \bar{y}_t) + Dg(t, \bar{y}_t)(\xi_t[\hat{\alpha}] + z_t[\hat{v}, \hat{z}_0]) \leq -\varepsilon, \\ c(t, \bar{u}_t, \bar{y}_t) + Dc(t, \bar{u}_t, \bar{y}_t)(\hat{v}_t, \xi_t[\hat{\alpha}] + z_t[\hat{v}, \hat{z}_0]) \leq -\varepsilon, \\ \hat{\alpha}_t \geq \varepsilon. \end{cases}$$

We denote now by $\Lambda(\tilde{P}_N)$ the set of generalized Lagrange multipliers of problem (\tilde{P}_N) at $(\bar{u}, \alpha = 0, \bar{y})$. Following the proof of Lemma 3.10, we easily obtain that

$$(B.14) \quad \Lambda(\tilde{P}_N) = \{(\lambda, \gamma) \in \Lambda^N(\bar{u}, \bar{y}) \times L^\infty([0, T]; \mathbb{R}_+^k) : \gamma_t^i = H[p_t^\lambda](t, u_t^i, \bar{y}_t) - H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t), \text{ for } i = 1, \dots, N, \text{ for a.a. } t\},$$

where $\Lambda^N(\bar{u}, \bar{y})$ is defined by (3.5) and γ is associated with the constraint $\alpha \geq 0$.

LEMMA B.5. *Let $N \in \mathbb{N}$; all multipliers of $\Lambda^N(\bar{u}, \bar{y})$ are nonsingular iff problem (\tilde{P}_N) is qualified.*

Proof. It is known that all multipliers of $\Lambda(\tilde{P}_N)$ are nonsingular iff problem (\tilde{P}_N) is qualified; see, e.g., [9, Proposition 3.16]. It follows from (B.14) that all multipliers of $\Lambda^N(\bar{u}, \bar{y})$ are nonsingular iff the multipliers of $\Lambda(\tilde{P}_N)$ are nonsingular. This proves the lemma. \square

As a corollary, we obtain that if problem (\tilde{P}_N) is qualified at stage N , it is also qualified at stage $N + 1$. Indeed, if none of the multipliers in $\Lambda^N(\bar{u}, \bar{y})$ are singular, a fortiori, none of the multipliers in $\Lambda^{N+1}(\bar{u}, \bar{y})$ are singular, since $\Lambda^{N+1}(\bar{u}, \bar{y}) \subset \Lambda^N(\bar{u}, \bar{y})$.

PROPOSITION B.6. *The set of singular Pontryagin multipliers is empty iff there exists $N \in \mathbb{N}$ such that problem (\tilde{P}_N) is qualified.*

Proof. Let $N \in \mathbb{N}$ be such that problem (\tilde{P}_N) is qualified. Then, all multipliers of $\Lambda^N(\bar{u}, \bar{y})$ are nonsingular, by Lemma B.5. Since $\Lambda_P(\bar{u}, \bar{y}) \subset \Lambda^N(\bar{u}, \bar{y})$, the Pontryagin multipliers are nonsingular.

Conversely, assume that for all $N \in \mathbb{N}$, problem (\tilde{P}_N) is not qualified. By Lemma B.5, we obtain a sequence of singular multipliers $(\lambda^N)_N$ which is such that for all $N, \lambda^N \in \Lambda^N(\bar{u}, \bar{y})$. Normalizing this sequence, we obtain with Lemma 3.5 the existence of a (nonzero) weak limit point in $\Lambda_P(\bar{u}, \bar{y})$, which is necessarily singular. \square

To conclude the proof, we still need a relaxation result, which makes a link between the Pontryagin linearization $\xi[u]$ and the linearization $\xi[\alpha]$.

LEMMA B.7. *Let $N \in \mathbb{N}$; assume that problem (\tilde{P}_N) is qualified. Then, there exists $A > 0$ such that for all $(\alpha, v, z_0) \in \mathcal{A}^N \times \mathcal{U} \times \mathbb{R}^n$ with $\|\alpha\|_\infty \leq A, \|v\|_\infty \leq A, |z_0| \leq A$, for all $\varepsilon > 0$, if α is uniformly positive, then there exists $(u, v', z'_0) \in \mathcal{U}_c \times \mathcal{U} \times \mathbb{R}^n$ such that*

$$(B.15) \quad D\Phi^E(\bar{y}_0, \bar{y}_T)(z_0, \xi_T[u] + z_T[v, z_0]) = D\Phi^E(\bar{y}_0, \bar{y}_T)(z_0, \xi_T[\alpha] + z_T[v, z_0]),$$

$$(B.16) \quad \|\xi[u] - \xi[\alpha] + z[v' - v, z_0 - z'_0]\|_\infty \leq \varepsilon.$$

Proof. We only give some elements of the proof. Note that this result is a variant of Proposition 3.8 and can be obtained with Dmitruk’s result [14, Theorem 3]. Let us define

$$(B.17) \quad h(t, u, y) := D_y f(t, \bar{u}_t, \bar{y}_t)y + f(t, u, \bar{y}_t) - f(t, \bar{u}_t, \bar{y}_t).$$

Then, for all $u \in \mathcal{U}_c$, $\xi[u]$ is the solution to

$$(B.18) \quad \dot{\xi}_t[u] = h(t, \xi_t[u], u_t), \quad \xi_0[u] = 0,$$

and $\xi[\alpha]$, where $\alpha \in \mathcal{A}^N$ and $\alpha \geq 0$, is the solution to the relaxed system associated with the dynamics g and the Castaing representation. Indeed,

$$(B.19) \quad \begin{aligned} \dot{\xi}_t[\alpha] &= D_y f(t, \bar{u}_t, \bar{y}_t)\xi_t[\alpha] + \sum_{i=1}^N \alpha_t^i (f(t, u_t^i, \bar{y}_t) - f(t, \bar{u}_t, \bar{y}_t)) \\ &= \left(1 - \sum_{i=1}^N \alpha_t^i\right) h(t, \bar{u}_t, \bar{y}_t) + \sum_{i=1}^N \alpha_t^i (h(t, u_t^i, \bar{y}_t) - h(t, \bar{u}_t, \bar{y}_t)). \quad \square \end{aligned}$$

Let us conclude the proof of Theorem B.3. Let us assume that the set of singular Pontryagin multipliers is empty; we already know by Proposition B.6 that there exists $N \in \mathbb{N}$ such that problem (\tilde{P}_N) is qualified. It remains to prove that the problem is P-qualified. Let $(\alpha^k, v^k, z_0^k)_{k=1, \dots, n_{\Phi^E} + 1}$ be such that

$$(B.20) \quad 0 \in \text{int} \left\{ \text{conv} \left[D\Phi^E(\bar{y}_0, \bar{y}_T)(z_0^k, z_T[v^k, z_0^k]) + \xi_T[\alpha^k], k = 1, \dots, n_{\Phi^E} + 1 \right] \right\}.$$

Let $(\hat{\alpha}, \hat{v}, \hat{z}_0)$ be such that (B.13) holds. By (B.12), if we replace (α^k, v^k, z_0^k) by $(\alpha^k + \delta\hat{\alpha}, v^k + \delta\hat{v}, z_0^k + \delta\hat{z}_0)$ for any $\delta > 0$, then (B.20) still holds. Moreover, (B.20) remains true if we multiply this family by a given positive constant. Therefore, since $\hat{\alpha}$ is uniformly positive, we may assume that the family $(\alpha^k, v^k, z_0^k)_{k=1, \dots, n_{\Phi^E} + 1}$ is bounded by A and such that for all $k = 1, \dots, n_{\Phi^E} + 1$, α^k is uniformly positive. Finally, we can apply Lemma B.7 to any convex combination of elements of the family. This proves the part of the P-qualification associated with equality constraints. Multiplying $(\hat{\alpha}, \hat{v}, \hat{z}_0)$ by a positive constant, we can assume that it is bounded by A and we can equally approximate it so that (B.4) holds and so that (B.5) holds (if the variable ε of Lemma B.7 is chosen sufficiently small). We have proved that the problem was P-qualified.

Appendix C. An example about Pontryagin’s principle. We give here an example where there exists a multiplier such that inequality (2.24) holds for all $u \in U(t)$, but not for all u in $\tilde{U}(t) := \{u \in \mathbb{R}^m : c(t, u, \bar{y}_t) \leq 0\}$. Indeed, $U(t) \subset \tilde{U}(t)$ but the inclusion may be strict. Consider the following problem:

$$(C.1) \quad \min y_T, \text{ subject to: } \dot{y}_t = u_t, \quad y_0 = 0, \quad u_t \geq -y_t, \text{ for a.a. } t.$$

The optimal trajectory is $(\bar{u}, \bar{y}) = (0, 0)$. The problem being qualified, there exists a normal Lagrange multiplier which is determined by ν . Since the augmented Hamiltonian is stationary, we obtain that for a.a. t , $p_t^\nu = \nu_t$, and therefore,

$$(C.2) \quad -\dot{p}_t^\nu = -p_t^\nu, \quad p_T^\nu = 1,$$

i.e., $p_t = \nu_t = e^{-(T-t)} > 0$. We have: $\tilde{U}(t) = U(t) = \mathbb{R}_+$ and the Hamiltonian pu is minimized for a.a. t by $\bar{u}_t = 0$ since $p_t > 0$.

Now let us consider a variant of this problem. We replace the previous mixed constraint by $\psi(u_t) \geq -y_t$, where ψ is a smooth function such that

$$(C.3) \quad \begin{cases} \forall u \geq 0, \psi(u) = u, \\ \forall u < 0, \psi(u) \leq 0 \text{ and } \psi(u) = 0 \iff u = -1. \end{cases}$$

For $y^0 = 0$, $(0, 0)$ remains a feasible trajectory, since $\tilde{U}(t) = \mathbb{R}_+ \cup \{-1\}$. In this case, $U(t) = \mathbb{R}_+$. Let us check that $(0, 0)$ is still an optimal solution. Let us suppose that there exist a feasible trajectory (u, y) which is such that $y_T < 0$. Let us suppose that y reaches its minimum at a time t . Necessarily, $y_t < 0$ and $t > 0$. Let $0 < \delta < t$ be such that for all $s \in [t - \delta, t]$, $y_s < y_t/2$. The mixed constraint imposes that

$$(C.4) \quad u_s \geq -y_s \geq -y_t/2 \quad \forall s \in [t - \delta, t].$$

Thus, $y_t \geq y_{t-\delta} - \delta y_t/2 > y_t$, which is a contradiction. We have proved that $(0, 0)$ is an optimal solution, and the multiplier and costate remain unchanged. However, the minimum of the Hamiltonian over $\tilde{U}(t)$ is reached for all t at $u = -1 \neq \bar{u}_t$.

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